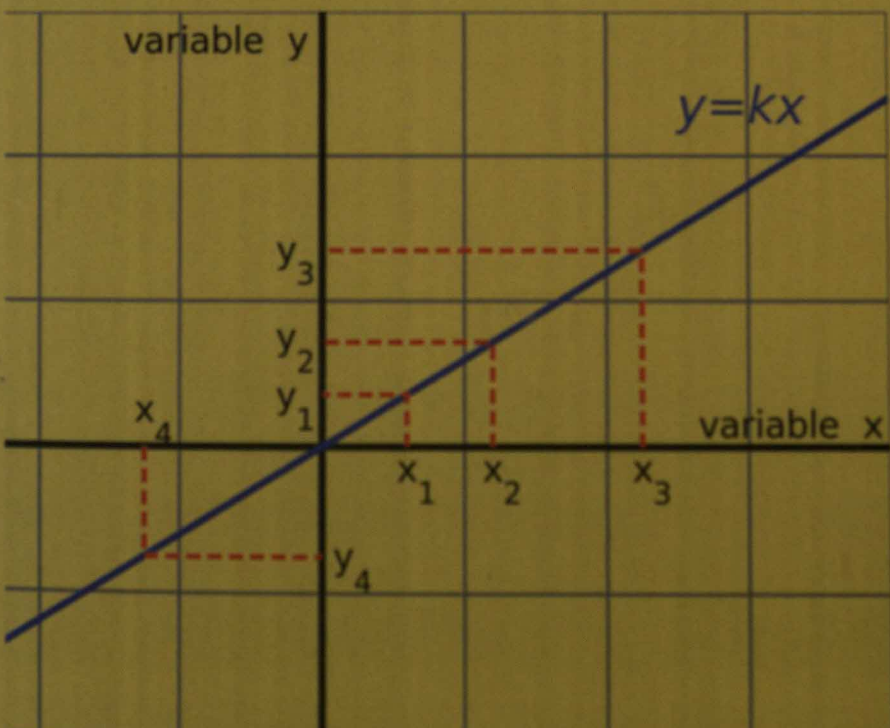
EASTERN

Differential Calculus for Degree

Mardor Wanri Synrem



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EBH Publishers (India)
Guwahati-1

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ISBN : 978 93 88881 25 8

First Published in 2020 by
EBH Publishers (India)
an imprint of Eastern Book House
136, M.L. Nehru Road, Panbazar
Guwahati- 781 001, Assam

Phone : +91 361 2513876, 2519231, 92070 45352
Fax : +91 361 2519231
Email : easternbookhouse@gmail.com.
www.easternbookhouse.in
Printed in India

Preface

This book is written in accordance with the new syllabus for the B. A. And B. Sc course of the North Eastern Hill University and is intended to be used as a text book by the students of other universities as well. It is hoped the Post Graduate and Engineering students also will find the book very helpful.

In this book each chapter has been written elaborately with due care without going into unnecessary details but is written in very simple language easily understandable to the students. Great care has been taken in writing the proofs of the theorems to make them as simple, clear and lucid as possible. Numerous worked out examples will help the students to understand the theory and concepts so that they can get clear idea and can easily solve other problems.

Finally, I take this opportunity to express sincere thanks to the publisher of this book, for the special care he has taken to bring out this edition of the book. Last but not the least I must not fail to mention my sincerest appreciation and thanks to my beloved wife, daughter, my distinguished colleagues and friends without whose inspiration, I would not be able to bring out this book in the present form. I would acknowledge any suggestions for improvements of the book in future editions.

Shillong
Sept, 2019 — Mardor Wanri Synrem

Contents

1

Set Theory

Introduction

As students have already done the topic of sets in their lower classes, it is expected that the students will take utmost interest while doing this chapter though with wider range.

Definitions:

1.1 Set

A set is a collection of well defined and distinct object of our perception or thought. The word "well defined" means a rule to be given with the help of which we should readily be able to say whether a particular object 'belongs to' the set or not. The word 'distinct' imply that if the objects of the collection be named and in doing so the number of objects will not increase.

Set are usually denoted by the capital letters of the English alphabet, say A, B, C, X, Y, Z, and so on.

Examples:

(i) The collection of all natural numbers is a set, denoted by $\mathbb{N}$

(ii) The collection of all integers is a set, denoted by $\mathbb{Z}$

(iii) The collection of most talented writers in India, is not a set.

Some standard sets Notation:

$\mathbb{N}$ = Set of all natural numbers

$\mathbb{Z}$= Set of all integers

IR = Set of all real numbers

Q = Set of all rational numbers

ℂ = Set of all complex numbers

W = Set of all whole numbers

1.2 Elements

The objects which constitute the set are called elements of the set. These are also known as members of the set.

The elements are usually denoted by small letters of English alphabets, say a, b, c, x, y, z,............

Given a set say S and an object say x, then one and only one of the following statements is true.

(i) x is in the collection S

(ii) x is not in the collection S

If x is in the collection S, then it is read as "x belongs to S" or "x is an element of s" or "x is in S" and is represented symbolically by $x \in S$. If x is not in the collection S, then it is read as "x does not belong to S" or "x is not an element of S" or "x is not in S" and is represnted symbolically by $x \notin S$.

Examples:

Let S: {0, 2, 4 , 6, 8, 10} then

(i) $6 \in S$ (ii) $3 \notin S$

1.3 Representation of a set

There are two types of representation of a set

(i) Roster or Tabulation Method. In this method, the set is represented by listing all its elements, seperating the elements by commas and enclosing them in curly brackets.

(ii) Defining Property Method. In this method, the set is represented by sepecifying the common property of the elements. In this case the set S is denoted as

S : {x : P(x) is true}

Here 'x' stands for 'an arbitrary element of the set and ':' stands for 'such that' and P(x) stands for "Common Property"

This form of the set is known as set builder form.

Exmaples:

(1) Let S be the set of odd natural numbers less than 10. Then Roster form of S is

S : {1, 3, 5, 7, 9}

and Set Builder form of S is

S = {x:x is an odd natural number and x < 10}

(2) Let S be the set of all letters of the word "MATHEMATICS". Then

Roster form of S is S = {A, E, I, C, H, M, S, T}

Set Builder form of S is S = {x : x is a letter of the word MATHEMATICS}

Illustrative Exmaples

Example 1. Explain the difference between a collection and a set. Justify your answer.

Solution: Each set is a collection but collection may not be a set as only well defined collection is a set.

Examples 2. Write the set of all positve integers whose cube is odd.

Solution: The elements of the required set are not even [Since cube of an even integer is an even integer]

Moreover, the cube of a positive odd integer is a positive odd integer i.e. the elements of the required set are all positive integers.

Hence, the required set in set builder form is

$\{2k + 1 : k \geq 0, k \in \mathbb{Z}\}$

Example 3. Write the set {x : x is a positive integer and $x^2 < 30$} in the roster form.

Solution: The squares of positive integers which are less than 30 are 1, 2, 3, 4, 5.

Hence the given set iin roster form is

{1, 2, 3, 4, 5}

Example 4. Match each of the sets on the left in the roster form with the same set on the right describe in the set builder form

(i)	{0}	(a)	{x : x is a letter of the word LITTLE}
(ii)	{1, 3, 5, 7, 9}	(b)	{x : x is a natural number and is a divisor of 6}

(iii) {L, T, T, E} (c) {x : x is an integer and x + 2 = 2}

(iv) {2, 3} (d) {x : x is an odd natural number less than 10}

1.4 Finite and Infinite Sets

(a) Finite Set

A Set a said to be finite if it consists of only finite number of elements

Examples: {1, 2, 3, 4}; {a, e, i, o, u} are finite sets

(b) Infinite Set

A set is said to be infinite if it consists of infinite number of elements

Examples: {1, 2, 3, 4,........}; {2, 4, 6, 8,.......} are infinite sets

(c) Cardinal Number

In a finite set the number of the elements it consists say 'n' is called the cardinal number or the order of this finite set and is denoted by n (S)

1.5 The Empty Set

A set which does not contain any element is called the empty set. It is also called the 'null' or 'void' set. The empty set is denoted by ϕ.

Examples:

(i) Let A be the collection of all those integer whose square is negative or less than zero.

Then obviously A is an empty set since the square of any integer cannot be negative or less than zero.

(ii) B = {x : x is a positive integer < 1}

Obviously B is an empty set since no positive integer is less than 1.

1.6 The Singleton Set

A set having only one element is called a Singleton Set.

Example: {0} is a Singleton Set

1.7 Order of a Finite Set

The number of distinct elements in a finite set S is called the order of the set S and is denoted by 0 (S)

If the order of the set is zero the set is empty.

If the order of the set is one, the set is Singleton.

Remark: The order of an infinite set is never defined.

1.8 Equal and Equivalent Sets

(a) Equal Sets

Two Sets A and B are said to be equal if they have same elements.

If two sets A and B are equal, it is written as A = B and if the two sets A and B are not equal, it is written as $A \neq B$.

Examples:

(1) Let A = {1, 2, 3, 4} and B = {2, 4, 1, 3}

Then A = B

(2) Let A = {1, 2, 3, 4} and B = {1, 2, 3, 5}

Then $A \neq B$

(b) Equivalent Sets

Two finite sets A and B are said to be equivalent sets if they have same number of elements.

If two sets A and B are equivalent, it is written as $A \approx B$.

Example:

Let A = {1, 2, 3, 4} and B = {a, b, c, d}

Then $A \approx B$ [Since A and B have four elements each]

1.9 Subsets

Let A and B be two sets. Then, the set A is said to be a subset of B if every element of A is also an element of B.

If A is a subset of B, we write $A \subset B$.

When A is a subset of B, it means that "A is contained in B" or "B contains A". Here B is called a super set of A and is written as $B \supset A$.

Remark:

1. Every set is a subset of itself i.e. $A \subset A$

2. Null set or Empty set is a subset of every set i.e. $\phi \subset A$ for all A.

If A is not a subset of B, we write $A \not\subset B$

Result: If the number of elements in a finite set is 'n', then the total number of subsets is 2^n.

Example:

Let A = {1, 2, 3, 4}, B = {2, 4} and C = {3, 4}

(i) Is $B \subset A$? (ii) Is $C \subset B$? (iii) Is $B \subset C$? (iv) Is $C \subset A$?

Solution:

(i) Yes, since each element of B is in A

(ii) No, since $3 \in C$ but $3 \notin B$

(iii) No, since $2 \in B$ but $2 \notin C$

(iv) Yes, since each element of C is in A

1.10 Comparable Sets

Two sets A and B are said to be comparable iff either $A \subset B$ or $B \subset A$

Another Definition of Equal Sets. Two sets A and B are said to be equal (i.e. A = B) iff A is a subset of B and B is a subset of A.

1.11 Proper Subset

If $A \subset B$ and $A \neq B$, then we say that A is a proper subset of B.

A non empty set A is said to be a proper subset of B if there is at least one element of B, which is not in A.

If the subset is not proper it is called improper subset. $A \subset A$ and $\phi \subset A$ are improper subsets.

1.12 Power Set

The collection of all possible subsets of a given finite set A, is called the power set of A and is denoted by P(A).

Examples: Let A = {1, 2, 3}

Then P(A) = {ϕ, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}}

1.13 Universal Set

The set for which each of the sets under consideration are its subsets, is called a Universal Set, which is generally denoted by U or X.

Note: The Universal Set is not unique

Examples:

(i) For the set of all integers $\mathbb{Z}$, the Universal Set can be the set Q of rational number or IR, the set of real numbers.

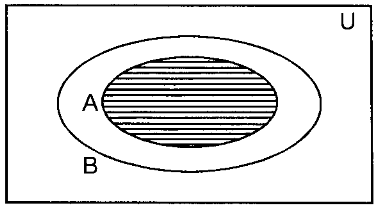
(ii) For (a) Set of all acute - angled triangles

(b) Set of all obstuse - angled triangles

(c) Set of all right - angled triangles

the universal set is the "Set of all triangles"

1.14 Euler-Venn Diagrams



"Venn diagrams' were named after John Venn (1834-1934). Many of the properties of sets can be verified with the help of these diagrams.

With the help of our geometric intuitions we talk of the universal set as the region enclosed by a rectangle and its subsets as regions enclosed by one or more than one closed curves.

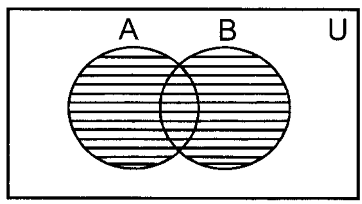
If A and B are two sets such that every element of A is also an element of B. Then we say that 'A is a subset of B' written as $A \subset B$ and this is shown by Venn diagram in the adjoining figure.

1.15 Operations on Sets

(a) Union of two sets:

The union of two sets A and B is the collection of all those elements which either belong to A or to B or to both A and B. It is denoted by $A \cup B$.

The common elements are to be taken only once.



Shaded region is $A \cup B$

Symbolically. $A \cup B = \{x : x \in A \text{ or } x \in B\}$

It is shown in the above fig by Venn diagram.

Examples:

(I) Let $A = \{a, b, c\}$, $B = \{a, e, i, o, u\}$

Then $A \cup B = \{a, b, c, e, i, o, u\}$

(II) Let $A = \{x : x \text{ is an even integer} \leq 10\}$

$B = \{ x : x \text{ is an integer and } 0 < x \leq 1\}$

Then $A \cup B = \{x : x \text{ is an integer, } 0 < x \leq 10\}$

Remark:

(i) $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$

(ii) $x \notin A \cup B \Leftrightarrow x \notin A$ or $x \notin B$

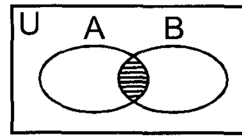
Entension: Let $A_1, A_2, ..., A_n$ be n sets then the union of all these sets is denoted by $\bigcup_{i=1}^{n} A_i$

(b) Intersection of Two Sets:

The intersection of two sets A and B is the collection of all those elements which belong to both A and B. It is denoted by $A \cap B$.

Symbolically. $A \cap B = \{x : x \in A \text{ and } x \in B\}$

It is shown in the adjoining figure by Venn diagram.



Shaded region is A∩B

Examples:

(I) Let A = {1, 2, 3, 4, 5, 6} B = (2, 4, 6}

Then $A \cap B = \{2, 4, 6\}$

(II) Let A = {x:x is a natural number}

B = {x:x is an integer}

Then $A \cap B$ = {x:x is a natural number|

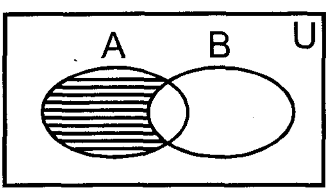
1.16 Difference of Two Sets

The difference of two sets A and B which is denoted by (A–B) is the collection of all those elements of A, which are not in the set B

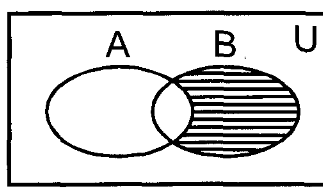
Symbolically: $A - B = \{x : x \in A, x \notin B\}$

Similarly $B - A = \{x : x \in B, x \notin A\}$

These are shown in the figs below by Venn diagrams



Shaded region is A–B



Shaded region is B–A

Remark:

(i) $x \in A - B \Leftrightarrow x \in A$ and $x \notin B$

(ii) $x \notin A - B \Leftrightarrow x \notin A$ and $x \in B$

Example:

Let A = {1, 2, 3, 4, 5, 6} B = {1, 3, 5}

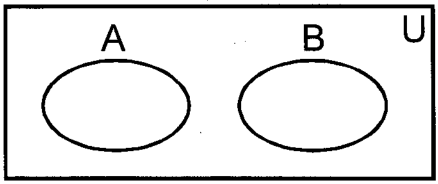
Then A–B = {2, 4, 6} and B–A = ϕ

1.17 Disjoint Sets

Two sets A and B are disjoint sets when they have no common elements

Thus if $A \cap B = \phi$, we say that A and B are disjont.

This is shown in the following Venn diagram.



Example:

Let A = {1, 3, 5, 7,.......} B= {2, 4 , 6, 8,......}

Then $A \cap B = \phi$. Thus A and B are disjoint sets

Notes:

(i) $\phi \subset A \; \forall A$ i.e. null set is contained in every subset

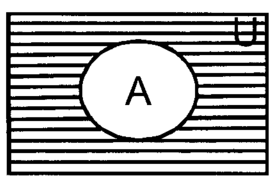
(ii) $\phi \cap A = \phi$ i.e. null set is disjoint from every subset.

1.18 Complement of a Set

Let U be the universal set and A be any subset of U .Then the complement of the set A is the collection of all those emelemts of U which are not in the set A. It is denoted by U–A or A^c or A'.

Symbolically, $A^c = \{x : x \in U \text{ and } x \notin A\}$

It is shown in the adjoning Venn digram



Shaded region is A^c

Remark:

(i) $x \in A^c \Leftrightarrow x \notin A$

(ii) $x \in A \Leftrightarrow x \notin A^c$

Remark:

(i) $A \cup A^c = U$ (ii) $A \cap A^c = \phi$

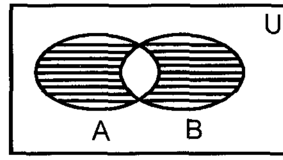
Example:

Let U = {1, 2, 3,..., 10} and A = {2, 4, 6, 8, 10}

Then A^c = {1, 3, 5, 7, 9}

1.19 Symmetric Difference of Sets

Let A and B be two sets. Then the set (A–B) ∪ (B–A) is called the symmetric difference of A and B which is denoted by $A \Delta B$.



Symbolically. $A \Delta B = \{x : x\ (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\}$

It is shown in the adjoining figure by Venn diagram.

1.20 Properties

(a) (i) $A \cup \phi = A$ (ii) $A \cap U = A$ (Identity Laws)

(iii) $A \cup A = A$ (iv) $A \cap A = A$ (Idempotent Laws)

(v) $A \cup B = B \cup A$ (vi) $A \cap B = B \cap A$ (Commulative Laws)

Proof: (Left as exercise)

(b) (i) $A \cup (B \cup C) = (A \cup B) \cup C$ (Associative Laws)

(ii) $A \cap (B \cap C) = (A \cap B) \cap C$

(iii) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (Distributive Laws)

(iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Proof: (i) Let $x \in A \cup (B \cup C)$

Then $x \in A \cup (B \cup C) \Leftrightarrow x \in A$ or $x \in B \cup C$ [by defn]

$\Leftrightarrow x \in A$ or ($x \in B$ or $x \in C$) [by defn]

$\Leftrightarrow$ ($x \in A$ or $x \in B$) or $x \in C$ [by Asso. Law]

$\Leftrightarrow x \in A \cup B$ or $x \in C$ [by defn]

$\Leftrightarrow x \in (A \cup B) \cup C$ [by defn]

Hence $A \cup (B \cup C) = (A \cup B) \cup C$

(ii) Let $x \in A \cap (B \cap C)$

Then $x \in A \cap (B \cap C) \Leftrightarrow x \in A$ and $x \in B \cap C$ [by defn]

$\Leftrightarrow x \in A$ and ($x \in B$ and $x \in C$ [by defn]

$\Leftrightarrow$ ($x \in A$ and $x \in B$) and $x \in C$ [by Ass. Law]

$\Leftrightarrow x \in A \cap B$ and $x \in C$ [by defn]

$\Leftrightarrow x \in (A \cap B) \cap C$ [by defn]

Hence $A \cap (B \cap C) = (A \cap B) \cap C$

(iii) Let $x \in A \cup (B \cap C)$

Then $x \in A \cup (B \cap C) \Leftrightarrow x \in A$ or $x \in B \cap C$ [by defn]

$\Leftrightarrow x \in A$ or ($x \in B$ and $x \in C$) [by defn]

$\Leftrightarrow$ ($x \in A$ or $x \in B$) and ($x \in A$ or $x \in C$)

$\Leftrightarrow x \in A \cup B$ and $x \in A \cup C$ [by defn]

$\Leftrightarrow x \in (A \cup B) \cap (A \cup C)$ [by defn]

Hence $A \cup (B \cap C) = (A \cup B) \cap (AC)$

(iv) Let $x \in A \cap (B \cup C)$

Then $x \in A \cap (B \cup C)$ $\Leftrightarrow x \in A$ and $x \in B \cup C$ [by defn]

$\Leftrightarrow x \in A$ and ($x \in B$) or $x \in C$) [by defn]

$\Leftrightarrow$ ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$)

$\Leftrightarrow x \in A \cap B$ or $x \in A \cap C$) [by defn]

$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$ [by defn]

Hence $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(c) (i) $(A^c)^c = A$ (Involution Law)

(ii) $A \cup A^c = \cup$, where $\cup$ is the universal set (Complement Law)

(iii) $A \cap A^c = \phi$ (Complement Law)

(iv) $(A \cup B)^c = A^c \cap B^c$ (De-Morgan Law)

(v) $(A \cap B)^c = A^c \cup B^c$ (De-Morgan Law)

Proof: (i) Let $x \in (A^c)^c$

Then $x \in (A^c)^c$ $\Leftrightarrow x \notin A^c$ [by defn]

$\Leftrightarrow x \in A$ [by defn]

Hence $(A^c)^c = A$

(ii) Let $x \in A \cup A^c$

Then $x \in A \cup A^c$ $\Leftrightarrow x \in A$ or $x \in A^c$ [by defn]

$\Leftrightarrow x \in A$ or $x \in \cup - A$, where $\cup$ is the universal set

$\Leftrightarrow x \in A$ or ($x \in \cup$, $x \notin A$)

$\Leftrightarrow x \in \cup$

Hence $A \cup A^c = \cup$

(iii) Let $x \in A \cap A^c$

Then $x \in A \cap A^c$ $\Leftrightarrow x \in A$ and $x \in A^c$ [by defn]

$\Leftrightarrow x \in A$ and $x \notin A$ [by defn]

Hence $A \cap Ac = \phi$

(iv) Let $x \in (A \cup B)^c$

Then $x \in (A \cup B)^c \Leftrightarrow x \notin A \cup B$ [by defn]

$\Leftrightarrow x \notin A$ and $x \notin B$ [by defn]

$\Leftrightarrow x \in A^c$ and $x \in B^c$ [by defn]

$\Leftrightarrow x \in A^c \cap B^c$

Hence $(A \cup B)^c = A^c \cap B^c$

(v) Let $x \in (A \cap B)^c$

Then $x \in (A \cap B)^c \Leftrightarrow A \notin A \cap B$ [by defn]

$\Leftrightarrow A \notin A$ or $x \notin B$ [by defn]

$\Leftrightarrow x \in A^c$ or $x \in B^c$ [by defn]

$\Leftrightarrow x \in A^c \cup B^c$ [by defn]

Hence $(A \cup B)^c = A^c \cup B^c$

Illustrative Examples

Example 1. Explain the difference between a collection and a set. Justify your answer.

Solution: Each set is a collection but collection may not be a set as only well defined collection is a set.

Example 2. Write the set off all positive integer whose cube is odd.

Solution: The elements of the required set are not even [Since cube of an even integer is an even integer]

Moreover, the cube of a positive odd integer is a positive odd integer i.e. the elements of the required set are all odd positive integers.

Hence the required set in set builder form as

$$\{2k + 1 : k \geq 0, k \in \mathbb{Z}\}$$

Example 3. Write the set $\{x : x$ is a positive integer and $x^2 < 30\}$ in the roster form.

Solution: The square of positive integers which are less than 30 are 1, 2, 3, 4, 5

Hence the given set in roster form is

$$\{1, 2, 3, 4, 5\}$$

Example 4. State which of the following sets are finite and which are infinite:

(i) $A = \{x : x \in \mathbb{N}$ and $x^2-3x+2 = 0\}$

(ii) $B = \{x : x \in IN \text{ and } x^2 = 9\}$

(iii) $C = \{x : x \in IN \text{ and x is even}\}$

(iv) $D = \{x : x \in IN \text{ and } 2x-3 = 0\}$

Solution: (i) $A = \{1, 2\}$

$[\because x^2-3x+2=0 \Rightarrow (x-1)(x-2) = 0 \Rightarrow x=1, 2]$

Hence A is finite set.

(ii) $B = \{3\}$

$[\because x^2 = 9 \Rightarrow x = \pm 3 \text{ But } 3 \in IN]$

Hence B is a finite set

(iii) $C = \{2, 4, 6, 8, ...\}$

Hence C is an infinite set

(iv) $D=\phi$ $\quad [2x-3=0 \Rightarrow x = \frac{3}{2} \notin IN]$

Hence D is a finite set

Example 5. Which of the following are empty (null) set?

(i) $A = \{x : x \in IN \text{ and } x^2 < 0]$

(ii) $B = \{x : x \in \mathbb{Z} \text{ and } x^2 + 3x - 4 = 0\}$

Solution: (i) A is a null set since square of any natural number cannot be negative.

(ii) $B = \{-4, 1\} \neq \phi$

$[x^2 + 3x - 4 = 0 \Rightarrow (x + 4)(x - 1) = 0 \Rightarrow x = -4, 1]$

B is not a null set

Example 6. For any two sets P and Q, prove that $P-Q=P \cap Q^c = Q^c-P^c$ where P^c is the compliment of P (NEHU, 2001)

Solution: Let $x \in P-Q$. Then

$x \in P-Q \Leftrightarrow x \in P$ and $x \notin Q$ [by defn]

$\Leftrightarrow x \in P$ and $x \in Q^c$ [by defn]

$\Leftrightarrow x \in P \cap Q^c$ [by defn]

$\therefore \quad P-Q = P \cap Q^c$ ....(i)

Also $x \in P-Q \Leftrightarrow x \in P$ and $x \notin Q$ [by defn]

$\Leftrightarrow x \notin P^c$ and $x \in Q^c$ [by defn]

$\Leftrightarrow x \in Q^c - P^c$[by defn]

$\therefore$ $P-Q = Q^c-P^c$ ....(ii)

By (i) and (ii) $P-Q = P\cap Q^c = Q^c-P^c$

Example 7. For any two sets A and B, prove that $A\subseteq B \Leftrightarrow A\cap B=A$

(NEHU, 2002)

Solution: Suppose $A\subseteq B$ and let $x\in A\cap B$

Then $x\in A\cap B \Leftrightarrow x\in A$ and $x\in B$ [by defn]

$\Leftrightarrow x\in A$ [as $A\subseteq B$]

$\therefore$ $A\cap B=A$

Conversely suppose $A\cap B = A$ and let $x\in A$

Then $x\in A \Rightarrow x\in A\cap B$ [since $A\cap B=A$]

$\Rightarrow x\in A$ and $x\in B$ [by defn]

$\Rightarrow x\in B$

$\therefore$ $x\in A \Rightarrow x\in B$

Hence $A\subseteq B$ [by defn]

Therefore $A\subseteq B \Leftrightarrow A\cap B = A$

Example 8. Prove that $A - (B\cup C) = (A-B)\cap(A-C)$

(NEHU, 2003, 2005, 2010, 2012, 2016)

Solution: Let $x\in A - (B\cup C)$

Then $x\in A-(B\cup C)$ $\Leftrightarrow x\in A$ and $x\notin B\cup C$ [by defn]

$\Leftrightarrow x\in A$ and $(x\notin B$ and $x\notin C)$

$\Leftrightarrow (x\in A$ and $x\notin B)$ and $(x\in A$ and $x\notin C)$

$\Leftrightarrow x\in(A-B)$ and $x\in (A-C)$ [by defn]

$\Leftrightarrow x\in(A-B) \cap (A-C)$

$\therefore$ $A - (B\cup C) = (A-B) \cap (A-C)$

Example 9. If A and B be any sets, prove that $A\cap(A^C\cup B) = A\cap B$ (NEHU 2004)

Solution: Let $x\in A\cap(A^C \cup B)$

Then $x\in A \cap (A^C \cup B) \Leftrightarrow x\in A$ and $x\in(A^C \cup B)$ [by defn]

$\Leftrightarrow x\in A$ and $(x\in A^C$ or $x\in B)$ [by defn]

$\Leftrightarrow x\in A$ and $(x\notin A^C$ or $x\in B)$

$\Leftrightarrow x\in A$ and $x\in B$

$\Leftrightarrow x\in A\cap B$ [by defn]

$\therefore$ $A\cap(A^C \cup B) = A\cap B$

Example 10. State and Prove De. Morgan's Laws. (NEHU 2013, 2016)

or, Show that the complement of the union of any number of sets is the intersection of their complements. (NEHU, 2006, 2007)

Solution: Refer to the definition of De Morgan's Laws (left as exercise for students)

Example 11. For any two sets A and B, prove that $A-B = \phi$ if and only if $A \subseteq B$ (NEHU, 2007 Pre Revised)

Solution: Let $A \subseteq B$ and $x \in A-B$

Then $x \in A-B \Leftrightarrow x \in A$ and $x \notin B$ [by defn]

But $x \notin B$ is absurd since $A \subseteq B$

$\therefore$ $A-B = \phi$

Hence if $A \subseteq B$, then $A-B = \phi$

Conversely suppose $A-B = \phi$ and let $x \in A$

Then $x \in A$ and $A-B = \phi \Rightarrow x \in B$ (by defn)

Hence $x \in A \Rightarrow x \in B \Rightarrow A \subseteq B$

Therefore $A-B = \phi$ if and only if $A \subseteq B$

Example 12. Let A, B, C be any three sets. Then prove that $(A \cup B) \cup C = A \cup (B \cup C)$ (NEHU, 2008 Pre Revised)

Solution: Refer to Associative Property.

Example 13. Let A={1, 2, 3, 4}, B={1, 2, 3} and C= {2,4}. Find all sets X such that -

(i) $X \subseteq B$ and $X \subseteq C$

(ii) $X \subseteq A$ and $X \not\subseteq B$ [NEHU, 2011]

Solution: A={1, 2, 3, 4} subsets of A are

X = ϕ, {1}, {2}, {3}, {4}, {1, 2}, {1, 3}, {1, 4} {2, 3}, {2, 4}, {3, 4}, {1, 2, 3}, {1, 2, 4}, {1, 3, 4}, {3, 4}, {2, 3, 4}, {1, 2, 3, 4}

B = {1, 2, 3}. Subsets of B are

X = ϕ, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3}

C = {2, 4} Subsets of C are

X = ϕ, {2}, {4}, {2, 4}

(i) X = ϕ, {2}

(ii) X = {4}, {1, 4}, {3, 4}, {2, 4}, {1, 2, 4}, {1, 3, 4}, {2, 3, 4}, {1, 2, 3, 4}

Example 14. Prove that for any 3 sets A, B, C

$$(A-C)\cap(B-C) = (A\cap B) - C \quad \text{(NEHU, 2012)}$$

Solution: Let $x\in(A-C)\cap(B-C)$

Then $x\in(A-C)\cap(B-C) \Leftrightarrow x\in(A-C)$ and $x\in(B-C)$

$\Leftrightarrow (x\in A$ and $x\notin C)$ and $(x\in B$ and $x\notin C)$ [by defn]

$\Leftrightarrow (x\in A$ and $x\in B)$ and $x\notin C$

$\Leftrightarrow x\in A\cap B$ and $x\notin C$

$\Leftrightarrow x\in(A\cap B) - C$ [by defn]

$\therefore (A-C) \cap (B-C) = (A\cap B) - C$

Example 15. Prove that for any two sets A, B

$$(A\cap B) \cup (A-B) = A$$

Solution: Let $x\in(A\cap B) \cup (A-B)$

Then $x\in(A\cap B) \cup (A-B) \Leftrightarrow x\in(A\cap B)$ or $x\in(A-B)$ [by defn]

$\Leftrightarrow (x\in A$ and $x\in B)$ or $(x\in A$ and $x\notin B)$ [by defn]

$\Leftrightarrow x\in A$ [since $x\in B$ and $x\notin B$ is absurd]

$\therefore (A\cap B) \cup (A-B) = A$

Example 16. Prove that $A-B=A \Leftrightarrow A\cap B=\phi$ where A and B are any two sets [NEHU, 2013]

Solution: Suppose $A-B = A$

Let $x\in A\cap B \Rightarrow x\in A$ and $x\in B$ [by defn]

$\Rightarrow x\in(A-B)$ and $x\in B$ [$\therefore A-B=A$]

$\Rightarrow x\in A$ and $x\notin B$ and $x\in B$

Since and $x\in B$ and $x\notin B$ cannot hold simultaneously

Hence this is absurd

$\therefore A\cap B=\phi$

Conversely suppose $A\cap B=\phi$ and let $x\in A-B$

Since $x\in A-B \Rightarrow x\in A$ and $x\notin B$

$\Rightarrow x\in A$ as $A\cap B=\phi$

$\therefore A-B = A$

Therefore $A-B = A \Leftrightarrow A\cap B=\phi$

Exercise 1.1

1. Which of the following are sets? Justify your answers
   (i) The collection of all months in a year beginning with the letter J
   (ii) The collection of most talentd writers of India
   (iii) A team of eleven best Cricket batsman of India
   (iv) The collection of difficult topics in Mathematics
   (v) The collection of novels written by Khushwant Singh
2. Write the following sets in Roster form:
   (i) A = {x : x is a natural number less than 10}
   (ii) B = {x : x is a two digit natural number such that the sum of its digits is 6}
   (iii) C = {x : x is a positive integer and $x^2<40$}
   (iv) D = {x : x is a letter of the word 'TRIGONOMETRY'}
   (v) E = {x : x is an integer whose cube is an even integer}
3. State which of the following sets are limite and which are finite:
   (i) $A = \{x : x \in \mathbb{N} \text{ and } (x-2)(x-3) = 0\}$
   (ii) $B = \{x : x \in \mathbb{Z} \text{ and } x^2 = 36\}$
   (iii) $C = \{x : x \in \mathbb{N} \text{ and } 2x+1 = 0\}$
   (iv) $D = \{x : x \in \mathbb{N}$ and x is a prime}
4. Which of the following are empty or null sets?
   (i) $A = \{x : 5<x<6, x \in \mathbb{N}\}$
   (ii) Set of even prime numbers
   (iii) $\{x : x^2 = 25$ and x is an even integer}
   (iv) $\{x : x \in \mathbb{N} \text{ and } x^2+1 = 0\}$
5. Are the following sets equal? Give reasons.
   (i) A = {2, 3}, B = {x : x is a root $x^2-5x+6=0$}
   (ii) $A = \{n : n \in \mathbb{Z} \text{ and } n^2 \leq 4\}$ $B = \{x : x \in \mathbb{R} \text{ and } x^2-3x+2=0\}$
   (iii) A = {x : x is a letter in the world "LOYAL"}
   B = {x : x is a letter in the world "ALLOY"}
   (iv) $A = \{x : x \in \mathbb{N}, x<3\}$ B = {1, 2} C = {3, 1}
   $D = \{x : x \in \mathbb{N}$, x is odd and x<5}
   E = {1, 2, 1} F = {1, 1, 3}

6. Let A = {1, 2} {3, 4 }, 5}. Which of the following statements and true and false? Give reasons.

   (i) $\{3, 4\} \subset A$ (ii) $\{3, 4\} \in A$ (iii) $\{\{3, 4\}\} \subset A$

   (iv) $1 \subset A$ (v) $\{1, 2, 5\} \subset A$ (vi) $\{1, 2, 5\} \in A$

   (vii) $\phi \in A$ (viii) $\{\phi\} \subset A$

7. Write down all possible subsets of the following

   (i) $\{\phi\}$ (ii) $\{1\}$ (iii) $\{1, 2, 3\}$ (iv) $\{1, \{1\}\}$

8. Write down the power set of the following:

   (i) $\{0\}$ (ii) $\{1, 2\}$ (iii) $\{a, b, c\}$

9. Prove that $A \subset \phi$ implies $A = \phi$

10. Let A, B and C be thre sets. If $A \subset B$ and $B \in C$, is it true that $A \subset C$? If not give an example.

11. Prove that $A \subset B, B \subset C \Rightarrow A \subset C$

12. Find the union and intersection of the following pair of sets

    (i) $A = \{1, 2, 3, 4\}$; $B = \{2, 3, 5\}$

    (ii) $A = \{a, e, i, o, u\}$; $B = \phi$

    (iii) $A = \{x : x$ is a natural number and $1 < x \leq 6\}$

    $B = \{x : x$ is a natural number and $6 < x \leq 10\}$

    (iv) $A = \{x : x \in \mathbb{Z}^-$ and $x^2 > 7\}$, $B = \{1, 2, 3\}$

    (v) $A = \{x : x \in \mathbb{Z}^-\}$ and $B = \{x : x \in \mathbb{Z}$ and $x < 0\}$

13. Let $A = \{x : x$ is a natural number$\}$

    $B = \{x : x$ is an even natural number$\}$

    $C = \{x : x$ is an odd natural number$\}$

    $D = \{x : x$ is a prime number$\}$

Find (i) $A \cap B$ (ii) $B \cap C$ (iii) $B \cap D$ (iv) $A \cap C$ (v) $A \cap D$

Find (i) $A - B$ (ii) $A - D$ (iii) $B - A$ (iv) $C - A$ (v) $D - B$

14. If $\mathbb{R}$ is the set of real numbers and Q is the Set of rational numbers, then what is $\mathbb{R} - Q$?

15. Which of the following are disjoint sets?

    (i) $\{1, 2, 3, 4, 5\}$ and $\{x : x$ is a natural nuber and $5 \leq x \leq 7\}$

    (ii) $\{a, e, i, o, u\}$ and $\{b, c, d, e, f\}$

    (iii) $\{x : x$ is an odd integer$\}$ and $\{x : x$ is an even integer$\}$

16. Let $\mathbb{N}$ be the universal set. Write down the complements of the following sets:

    (i) $\{x : x \in \mathbb{N}$ and x is odd$\}$

    (ii) $\{x : x \in \mathbb{N}$ and x is even$\}$

    (iii) $\{x : x$ is a prime number$\}$

    (iv) $\{x : x \in \mathbb{N}$ and $x=3n$ for some $n \in \mathbb{N}\}$

    (v) $\{x : x \in \mathbb{N}$ and x is a perfect square$\}$

    (vi) $\{x : x \in \mathbb{N}$ and $2x+5=11\}$

17. Prove that (i) $A \subset A \cup B$ (ii) $A \cap B \subset A$

18. If $A \cap B^c = \phi$, show that $A \subset B$

19. If A and B are any two sets, prove that

    (i) $A-B = A \cap B^c$ (ii) $(A-B) \cup B = A \cup B$

    (iii) $A-B = A-(A \cap B)$ (iv) $A \subset B \Leftrightarrow B^c \subset A^c$

20. If A, B and C are any three sets, then prove that:

    (i) $A \cap (B-C) = (A \cap B) - (A \cap C)$

    (ii) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$

    (iii) $(A \cup B) - C = (A-C) \cup (B-C)$

    (iv) $A \cap (B-C) = (A \cap B) - C$

    (v) $A \cap (B-A) = \phi$

    (vi) $(A-B) \cap (B-A) = \phi$

    (vii) $(A-B) \cap (A \cap B) = \phi$

    (viii) $\phi - A = \phi$

21. Show that $A \cap B = A \cap C$ need not imply $B=C$.

22. Prove that the number of subsets of a set with 'n' elements is 2^n.

Important Results:

If A, B and C are finite sets and $\cup$ is the universal set then

(i) $n(A \cup B) = n(A) + n(A \cap B)$

(ii) $n(A \cup B) = n(A) + n(B)$ if A, B are disjoint non empty sets

(iii) $n(A) = n(A-B) + n(A \cap B)$

(iv) $n(A \Delta B) =$ No. of elements which belong to exactly one of A or B

$= n((A-B) \cup (B-A))$

$= n((A-B) + n(B-A))$ [$\because$ A–B and B–A are disjoint] i.e. $n(A \Delta B) = (n(A) - n(A \cap B)) + n(B) - n(B \cap A)$

$= n(A) + n(B) - 2n(A \cap B)$ [$\because A \cap B = B \cap A$]

(v) $n(A \cap B^c) = n(A) - n(A \cap B)$

(vi) $n(B \cap A^c) = n(B) - n(A \cap B)$

(vii) $n(A \cup B) = n(A \cap B^c) + n(B \cap A^c) + n(A \cap B)$

(viii) $n(A^c \cup B^c) = n((A \cap B)^c) = n(\cup) - n(A \cap B)$

(ix) $n(A^c \cap B^c) = n((A \cup B)^c) = n(\cup) - n(A \cup B)$

(x) $n(A \cup B \cup C) = [n(A) + n(B) + n(C)] - [n(A \cap B) + n(A \cap C)$
$+ n(A \cap C) + n(B \cap C)] + n \ (A \cap B \cap C)$

Illustrative Examples

Example 1. If X and Y are two sets such that $n(X) = 17$, $n(Y) = 23$ and $n(X \cup Y) = 38$. Find $n(X \cap Y)$

Solution: We know that

$$n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$$

$\therefore 38 = 17 + 23 - n(X \cap Y)$

Hence $n(X \cap Y) = 40 - 38 = 2$

Example 2. If A and B be two sets containing 3 and 6 elements respectively. Find the maximum and minimum nunber of elements in $A \cup B$.

Soluition: We know that

$$n(A \cup B) = n(A) + n(B) - (A \cap B) \quad \text{..........} \quad (1)$$

Case 1: When $n(A \cap B)$ is minimum, then $n(A \cap B) = 0$

$\therefore n(A \cup B) = n(A) + n(B) = 3 + 6 = 9$

Hence maximum number of elements in $A \cup B$ is 9.

Case 2: When $n(A \cap B)$ is maximum, then $n(A \cap B) = 3$

$\therefore n(A \cup B) = n(A) + n(B) - n(A \cap B) = 3 + 6 - 3 = 6$

Hence minimum number of elements in $A \cup B$ is 6.

Example 3: Out of 20 number in a family 11 like to take coffee and 14 like to take tea. Assume that each one likes at least one of the two drinks. How many like

(i) both tea and coffee

(ii) only tea and not coffee

(iii) only coffee and not tea

Solution: Let T = Set of members who like tea

C = Set of members who like coffee

By question, n (T) =14 and n (C) = 11, n (T∪C) = 20

(i) Using n (T∪C) = n (T) + n (C) – n (T∩C) we get

n (T∩C) = n (T) + n (C) – n (T∪C)

= 14 + 11 – 20 = 5

∴ 5 members like both coffee and tea

(ii) n ($T \cap C^c$) = n (T) – n (T∩C)

= 14 – 5 = 9

∴ 9 members like only tea and not coffee

(iii) n ($C \cap T^c$) = n (C) – n (T∩C)

= 11 – 5 = 6

∴ 6 members like only coffee and not tea

Example 4. A survey report reveals that 59% of college students like tea whereas 72% like coffee. Find the possible range of the percentage of college students who like both tea and coffee. (NEHU, 2003)

Solution: Let T = Set of students who like tea

C = Set of students who like coffee

The n (T) = 59, n (C) = 72, n (C∪T) = 100

Using n (C∪T) = n(C) + n (T) – n(C∩T) we get

100 = 72 + 59 – n (C∩T)

ie n (C∩T) = 72 + 59 – 100 = 31

∴ Students who like both coffee and tea is 31%

Example 5: In a survey of 60 people, it was found that 25 people read newspaper A, 26 read rewspaper B, 26 read newspaper C, 9 read both A and C, 11 read both A and B, 8 read both B and C, 3 read all three newspaper. Find the number of people who read exactly one newspaper. (NEHU, 2004)

Solution: By question

n (A) = 25, n (B) = 26, n (C) = 26

n (A∩C) = 9, n (A∩B) = 11, n (B∩C) = 8, n (A∩B∩B) = 3

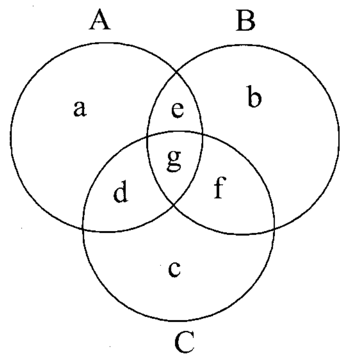
Using n (A∪B∪C) = n (A) + n (B) + n (C) – n (A∩B) – n (A∩C) – n (B∩C) + n (A∩B∩C) we get

$n(A \cup B \cup C) = 25 + 26 + 26 - 11 - 9 - 8 + 3$

$= 52$

The number of people reading at least one of the three newspaper per is 52

If a, b, c, d, e, f, g denote the nunber of elements in the respective regions, then by question,



$n(A) = a+e+d+g$

$n(B) = b+e+f+g$

$n(c) = c+d+f+g$

$n(A \cap B) = e+g,$

$n(A \cap C) = d+g,\ n(B \cap C) = f+g$

$n(A \cap B \cap C) = g$

Now, $n(A \cap B \cap C) = 3 \Rightarrow g = 3$

$n(A \cap B) = 11 \Rightarrow e+g = 11 \Rightarrow e = 8$

$n(A \cap C) = 9 \Rightarrow d+g = 9 \Rightarrow d = 6$

$n(B \cap C) = 8 \Rightarrow f+g = 8 \Rightarrow f = 5$

$n(A) = 25 \Rightarrow a+8+6+3 = 25 \Rightarrow a = 8$

$n(B) = 26 \Rightarrow b+8+5+3 = 26 \Rightarrow b = 10$

$n(C) = 26 \Rightarrow c+6+5+3 = 26 \Rightarrow c = 12$

No of people reading exactly one newspaper $= a+b+c$

$= 8+10+12$

$= 30$

Number of peole reading exactly one newspaper is 30.

Example 6. A college awarded 38 medals in Football, 15 in Basketball and 20 in Cricket. If 58 students recured all the medals such such that only three students got medals in all the three sports, how many received medals in exactly two of the three sports? (NEHU, 2005)

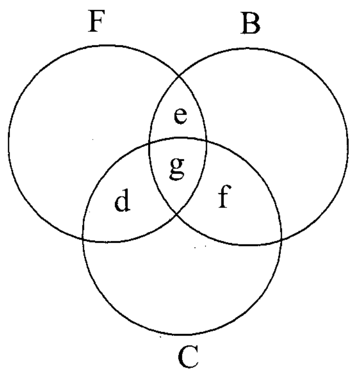
Solution: Let F = Set of students awarded medals in Football

B = Set of students awarded medals in Basketball

C = Set of students awarded medals in Cricket

By question, $n(F) = 38$, $n(B) = 15$, $n(C) = 20$, $n(F \cup B \cup C) = 58$, $n(F \cap B \cap C) = 3$

Using $n(F \cup B \cup C) = n(F) + n(B) + n(C) - n(F \cap B) - n(F \cap C) - n(B \cap C) + n(F \cap B \cap C)$ we get



$$58 = 38 + 15 + 20 - n(F \cap B) - n(F \cap C) - n(B \cap C) + - n(F \cap B \cap C)$$

$$\Rightarrow n(F \cap B) + n(F \cap C) + n(B \cap C) = 38 + 15 + 20 + 3 - 5\ 8 = 18$$

$$\Rightarrow e + g + f + g + d + g = 18$$

$$\Rightarrow e + f + d = 18 - 3g = 18 - 3 \times 3 = 18$$

$\therefore$ Number of students who received medals in exactly two of the three sports is 9.

Example 7. A Company wants to hire 25 programmers to handle systems programing jobs and 40 programmers for applications programming. Of those hired, ten will be expected to perform jobs of both types. How many programmers must be hired. (NEHU, 2008)

Solution: Let A = Set of programmers for programming jobs

B = Set of programmers for applications programming

By question, $n(A) = 25$, $n(B) = 40$, $n(A \cap B) = 10$

Using $n(A \cap B) = n(A) + n(B) - n(A \cap B)$ we get

$$n(A \cap B) = 25 + 40 - 10 = 55$$

$\therefore$ 55 progammers must be hired.

Example 8. In a survey of 600 students in a school, 150 students were found to be drinking tea, 225 drinking coffee and 100 were drinking both tea and coffee. Find how many students were drinking neither tea no coffee. (NEHU, 2011)

Solution: Let T = Set of students drinking tea

C = Set of students drinking Coffee

U = Set of students surveyed

By question, $n(U) = 600$, $n(T) = 150$, $n(C) = 225$, $n(C \cap T) = 100$

$$\text{Now } n(T^c \cap C^c) = n[(T \cup C)^c]$$
$$= n(U) - n(T \cup C)$$
$$= n(U) - [n(T) + n(C) - n(T \cap C)]$$
$$= 600 - [150 + 225 - 100]$$
$$= 600 - 275$$
$$= 325$$

$\therefore$ Number of students drinking neither coffee nor tea is 325.

Example 9. In a survey it was found that 21 persons liked product A, 26 liked product B and 29 liked product C. If 14 persons liked produicts A and B, 12 persons liked products C and A; 14 persons liked product B and C and 8 liked all the three products, find how many persons liked product C only.

(NEHU, 2012)

Solution: By question,

$n(A) = 21$, $n(B) = 26$, $n(C) = 29$

$n(A \cap B) = 14$, $n(C \cap A) = 12$, $n(B \cap C) = 14$, $n(A \cap B \cap C) = 8$

Required number of people who liked product C only

$= n(A^c \cap B^c \cap C)$

$= n((A \cup B)^c \cap C)$

$= n(C) - n((A \cup B) \cap C)$

$[n(A \cap B^c) = n(A) - n(A \cap B)]$

$= n(C) - n((A \cap C) \cup (B \cap C))$

$= n(C) - [n(A \cap C) + n(B \cap C) - n(A \cap B \cap C)]$

$= 29 - [12 + 14 - 8]$

$= 11$

Example 10. If 63% of persons like oranges where 76% like apples, then what can be said about the % of persons who like both oranges and apples?

(NEHU, 2013)

Solution: Same as example 4.

Exercise 1.2

1. There are 20 students in a Chemistry class and 30 students in a Physics class. Find the number of students which are either in Physics class or Chemistry class in the following cases:

   (i) The classes meet at the same time

   (ii) The two classes meet a different times and 10 students are enrolled in both the subjects. [Ans: (i) 50 (ii) 40]

2. In a survey of 400 students in a school, 100 were listed as drinking apple juice, 150 as drinking orange juice and 75 were listed as both drinking apple as well as orange juice. Find how many students were drinking neither apple juice nor orange juice. [Ans: 225]

3. Out of 500 car owners investigated, 400 owned Maruti cars and 200 owned Hyundai cars. 50 owned both Maruti and Hyundai Cars. Is this data correct? [Ans: No]

4. In a survey of 25 students, it was found that 15 had taken Mathematics, 12 had taken Physics and 11 had taken chemistry, 5 had taken Mathematics and Chemistry, 9 had taken Mathematics and Physics, 4 had taken Physics and Chemistry and 3 had taken all three subjects. Find the number of students that had taken:

   (i) only Chemistry

   (ii) only Mathematics

   (iii) only Physics

   (iv) Physics and Chemistry but not Mathematics

   (v) Mathematics and Physics but not Chemistry

   (vi) Only one of the subjects

   (vii) At least one of the three subjects

   (viii) None of the three subjects

   [Ans: (i) 5 (ii) 4 (iii) 2 (iv) 1 (v) 6 (vi) 11 (vii) 23 (viii) 2

5. There are 210 members in a club. 100 of them take tea and 65 take tea but not coffee. Each member takes tea or coffee

   (i) How many take coffee

   (ii) How many take coffee but not tea (NEHU, 2013)

   [Ans: (i) (ii)]

6. In a class of 50 students, 20 students play football and 16 students play hockey. It is found that 10 students play both the games. Use algebra of sets to find out the number of students who play neither football nor hockey.

(NEHU, 2014)

[Ans: 24]

7. In a group of 400 people, 250 can speak Hindi and 202 can speak English. How many can speak both Hindi and English. [Ans: 50]

8. In a group of 50 people, 35 speak Hindi, 25 speak both English and Hindi and all the people speak at least one of the two languages. How many people speak only English and not Hindi? How many people speak English. [NEHU, 2015] [Ans: 15, 40]

9. In a group of 70 people, 45 speak Hindi language and 33 speak English language and 10 speak neither Hindi nor English. How many can speak both English as well as Hindi? How many can speak only English language? [Ans: 18, 15]

10. Out of 80 students who secured first class marks in Mathematics or Physics, 50 obtained first class marks in Mathematics and 10 in both Physics and Mathematics. How many students secured first class marks in Physics only? [Ans: 30]

11. In a group of 50 people, 30 like to play cricket 25 like to play football and 32 like to play hockey. Assume that each one likes to play at least one of the three games. If 15 people like to play both cricket as well as football, 11 people like to play both football as well as hockey and 18 like to play both cricket as well as hocjey, then

   (i) how many like to play all the three games

   (ii) how many like to play only football

   (iii) how many like to play only hockey [Ans: (i) 7 (ii) 6 (iii) 10]

12. In a survey of 100 persons, it was found that 28 read magazine A, 30 read magazine B, 42 read magazine C, 8 read magazine A and B, 10 read magazines A and C, 5 read magazines B and C and 3 read all three magazines, Find

   (i) How many read none of the three magazines?

   (ii) How many read magazine C only? [Ans: (i) 20 (ii) 30]

13. In a survey of 60 people, it was found that 25 read newspaper H, 26 read newspaper T, 26 read newspaper I, 9 read both H and I, 11 read both H and T, 8 read both T and I, 3 read all three news paper. Find

   (i) the number of people who read at least one of the newspaper

(ii) the number of people who read exactly one newspaper.

(NEHU, 2004]

[Ans: (i) 52, (ii) 30]

14. In a class of 35 students, 15 study Economics, 22 study Business Studies and 14 study Advanced Accountancy. If 11 students study both Economics and Business Studies, 8 study both Business Studies and Advanced Accountancy and 5 study both Economics and Advanced Accountancy and 3 study all the three subjects. Find how many students of the class are not taking any of the three subjects?[Ans: 5]

15. A survey shows that 74% of Indians like apples, whereas 68% like oranges. What percentage of Indians like both apples and oranges?

[Ans: 42]

2

Relations and Functions

Cartesian Product of Sets

Definitions

2.1 Order pair

An orderd pair is a pair of entries in a specified order. It is also called ordered 2-tuple. The two entries are seperated by a comma and enclosed within brackets.

If A and B are any two sets, then by an ordered pair of elements we mean a pair (a, b) where $a \in A$, $b \in B$ in that order. The first element 'a' is called the first component and the second element 'b' is called the second component.

Two ordered pairs are equal if their corresponding components are equal. i.e. $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$.

The set of all ordered pair of elements (a, b); $a \in A$ and $b \in B$ is called the Cartesian product of two sets A and B and is denoted by $A \times B$

i.e. $A \times B = \{(a, b) ; a \in A, b \in B\}$

Example: Let $A = \{1, 2\}$ and $B = (3, 4, 5\}$

Then $A \times B = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5)\}$

Clearly, $n(A) = 2$, $n(B) = 3$, $n(A \times B) = 6$

Hence $n(A \times B) = 6 = 2 \times 3 = n(A) \times n(B)$

Remarks:

(i) $A \times B \neq B \times A$ if $A \neq B$

(ii) $A \times B = \phi$ if A or B or both A and B are empty.

2.2 Ordered Triplet:

It A, B, C are any three sets then by ordered triplet of elements we mean a triplet (a, b, c) where $a \in A$, $b \in B$, $c \in C$ in that order. It is also called ordered 3-tuple

The set of all ordered triplets (a, b, c); $a \in A$, $b \in B$, $c \in C$ is called the Cartesian triplet of three sets A, B and C is denoted by $A \times B \times C$

i.e $A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$

Similarly we can define the Cartesian product of n sets.

2.3 Ordered n-tuple:

If $A_1, A_2, \ldots. A_n$ are any n sets, then by ordered n-tuple we mean an n-tuple $(a_1, a_2, a_3, \ldots.. a_n)$; $a_1 \in A_1$, $a_2 \in A_2, \ldots.. a_n \in A_n$ in that order

The set of all ordered n-tuple $(a_1, a_2, \ldots.. a_n)$; $a_1 \in A_1$, $a_2 \in A_2, \ldots.. a_n \in A_n$ is called the Cartesian product of n sets $A_1, A_2, A_3, \ldots.. A_n$ denoted by $A_1 \times A_2 \times A_3 \times \ldots\ldots A_n$ or $\prod_{1=1}^{n} A_i$ where Π stands for the product.

Examples:

1. Find the values of a and b if (a + 2b, b + 1) = (3, 2)

Solution: (a + 2b, b + 1) = (3, 2) if and only if

a + 2b = 3 and b + 1= 2 (by defn)

$\therefore$ a + 2b = 3 and b = 2 – 1 = 1

$\therefore$ a = 3 – 2b = 3 – 2.1=1

$\therefore$ a = 1, b = 1

2. Let A = {1, 2, 3} and B {4, 5}. Find (i) $A \times B$ (ii) $B \times A$

Solution: (i) $A \times B$ = {(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)}

(ii) $B \times A$ = {(4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3)}

We see that $A \times B \neq B \times A$

2.4 Relation:

Let A and B be any two non empty sets. A relation R from a set A to a set B is a subset of $A \times B$ i.e $R \subseteq A \times B$.

A is called the domain of R and B the range or co-domain of R.

In particular, any subset $A \times A$ defines a relation in A.

Note: If $(a, b) \in R$, then we write aRb and it is read as "a is related to b by R."

If $(a, b) \notin R$, then we write $a \not R b$ and is read as "a is not related to b by R"

Example: Let A = {1, 2, 3, 4, 5, 6} and R be a relation in A given by R = {(a, b) : a–b = 2}

Then R = {(3, 1), (4, 2), (5, 3), (6, 4)}

Clearly $3R_1$, $4R_2$, $5R_3$, $6R_4$

But $1 \not R 3$, $2 \not R 4$, $3 \not R 5$

Note that $A \times A$ = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

We see that $R \subseteq A \times A$

2.5 Inverse of a Relation:

Let R be a relation from a set A to a set B and let (a, b) be member of the subset D of $A \times B$ corresponding to the relation R from A to B.

To the relation R from the set A to the set B, there corresponds a relation from the set B to the set A called the inverse of the relation R and is denoted by R^{-1} such that subset $B \times A$ corresponding to the relation R^{-1} is

$$\{(y, x) : (x, y) \in D\}$$

$$\text{i.e. } yR^{-1}x \Leftrightarrow xRy$$

Examples:

(i) The inverse of the relation "is a father of" in the set of all men is the relation "is the son of"

(ii) The inverse of the relation "is less than" in R is the relation "is greater than"

2.6 Types of Relations

(I) Empty Relation or Void Relation:

A relation R in a non empty set A is called an empty relation if no element of A is related to any element of A and we denote such relation by ϕ.

Thus $R = \phi \subseteq A \times A$

Example: Let A = {1, 2, 3, 4} and let R be a relation in A given by R = {(a, b) : a–b =6}

Clearly no element of $A \times A$ i.e. no $(a, b) \in R \subseteq A \times A$ satisfies the property a–b = 6

$\therefore$ R is an empty relation

(II) Universal Relation:

A relation R in a non empty set A is called universal relation if every element of A is related to every element of A.

Example: Let A = {1, 2, 3, 4}. Then

R = {(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2), (4, 3), (4, 4)}

is a universal relation in A.

(III) Identity Relation:

The relation $I_A = \{(a, a) : a \in A\}$ is called the identity relation on A

Example: Let A = {1, 2, 3, 4} then

$I_A = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$ is called the Identity relation on A

(IV) Reflexive Relation:

A relation R on a non empty set A is called a Reflexive Relation if aRa i.e $(a, a) \in R \; \forall a \in A$

Example: Let A = {1, 2, 3, 4} and R be a relation on A given by R = {(a, b) : a–b is even}

Clearly R = {(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (4, 2), (4, 4)}

We see that $(1, 1), (2, 2), (3, 3), (4, 4) \in R$ and hence R is a reflexive relation

(V) Symmetric Relation:

A relation R an a non empty set A is called Symmetric Relation if whenever aRb then bRa i.e if $(a, b) \in R \Rightarrow (b, a) \in R \; \forall a, b \in A$

Example : Let A = {1, 2, 3, 4} and R be a relation an A given by R = {(a, b) : a–b is even}

Then R = {(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (4, 2), (4, 4)}

Clearly, $(1, 3) \in R \Rightarrow (3, 1) \in R \Rightarrow (2, 4) \in R \Rightarrow (4, 2) \in R$

Hence R is symmetric

(VI) Transitive Relation:

A relation R on a non empty set A is called transitive relation if whenever aRb and bRc then aRc i.e if $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R \; \forall a, b, c \in A$

Example: Let A = {1, 2, 3, 4} and R be a relation an A given by R = {(a, b) : a–b is divisible by 3}

Then R = {(1, 1), (1, 4), (2, 2), (2, 5), (3, 3), (3, 6), (4, 1), (4, 4), (5, 2), (6, 3), (6, 6)}

We see that (1, 4)$\in$R and (4, 1)$\in$R $\Rightarrow$ (1, 1)$\in$R

(2, 5)$\in$R and (5, 2)$\in$R $\Rightarrow$ (2, 2)$\in$R

(3, 6)$\in$R and (6, 3)$\in$R $\Rightarrow$ (3, 3)$\in$R

Hence R is a transitive relation

2.7 Equivalence Relation

A relation R an a non empty set A is said to be an Equivalence Relation if and only if R is

(i) Reflexive (ii) Symmetric and (iii) Transitive

Example: Let R = $\{(a, b) : a, b \in \mathbb{Z}$ and $a+b$ is even$\}$

Then

(i) R is Reflexive: Clearly (a, a)$\in$R since for any $a\in\mathbb{Z}$, $a + a = 2a$ is even

(ii) R is Symmetric: We see that if (a, b)$\in$R, a, b$\in\mathbb{Z}$

Then a + b is even i.e $a + b = 2k$ for some $k\in\mathbb{Z}$

i.e $b + a = 2k$ for some $k\in\mathbb{Z}$

i.e (b, a) $\in$R

If (a, b)$\in$R, then (b, a)$\in$R

Hence R is Symmetric.

(iii) R is Transitive: Let (a, b)$\in$R and (b, c)$\in$R; a, b, c$\in\mathbb{Z}$

Then we see that (a, b)$\in$R $\Rightarrow$ a + b is even

i.e $a + b = 2l$, $l\in\mathbb{Z}$

Also (b, c)$\in$R $\Rightarrow$ b + c is even $\Rightarrow$ $b + c = 2m$, $m\in\mathbb{Z}$

Then $(a + b) + (b + c) = 2l + 2m$

$\Rightarrow a + c = 2l + 2m - 2b$

$\Rightarrow a + c = 2(l + m - b)$

$\Rightarrow$ a + c is even

$\Rightarrow$ (a, c)$\in$R

Hence R is transitive.

Since R is Reflexive, Symmetric and Transitive, therefore R is an Equivalence Relation.

2.8 Equivalence Class:

Let R be an equivalence relation defined an a non em pty set A and 'a' be a fixed element of A. Then the quivalence class of 'a' is the set of all those elements of A related to 'a' by R and is denoted by [a] i.e. $[a] = \{x \in A : (x, a) \in R, a \in A\}$

Example: On the set of integers $\mathbb{Z}$. Let R be a relation defined on $\mathbb{Z}$ as R = { a, b) : a, b$\in\mathbb{Z}$ and a–b is divisible by 3}

Clearly R is an equivalence relation on $\mathbb{Z}$

(Proof of this is same as in 2.7)

Now $[0] = \{x \in \mathbb{Z} : (x, 0) \in R, 0 \in \mathbb{Z}\}$
$= \{x \in \mathbb{Z} : x-0 \text{ is divisible by } 3\}$
$= \{0, \pm 3, \pm 6, \pm 9, \pm 12, \ldots\ldots..\}$

$[1] = \{x \in \mathbb{Z} : (x, 1) \in R, 1 \in \mathbb{Z}\}$
$= \{x \in \mathbb{Z} : x-1 \text{ is divisible by } 3\}$
$= \{0, -2, 4, -5, 7, -8, 10, -11, \ldots\ldots..\}$

$[2] = \{x \in \mathbb{Z} : (x, 2) \in R, 2 \in \mathbb{Z}\}$
$= \{x \in \mathbb{Z} : x-2 \text{ is divisible by } 3\}$
$= \{-1, 2, -4, 5, -7, 8, -10, 11, \ldots\ldots..\}$

$[3] = \{x \in \mathbb{Z} : (x, 3) \in R, 3 \in \mathbb{Z}\}$
$= \{x \in \mathbb{Z} : x-3 \text{ is divisible by } 3\}$
$= \{0, \pm 3, \pm 6, \pm 9, \pm 12, \ldots\ldots..\}$

$[4] = \{x \in \mathbb{Z} : (x, 4) \in R\}$
$= \{x \in \mathbb{Z} : x-4 \text{ is divisible by } 3\}$
$= \{0, -2, 4, -5, 7, -8, \ldots\ldots..\}$

Clearly we see that

$[0] = [3] = [6] = [9] = \ldots\ldots\ldots$
$[1] = [4] = [7] = [10] = \ldots\ldots.$
$[2] = [5] = [8] = [11] = \ldots\ldots.$

Note that $\mathbb{Z} = [0] \cup [1] \cup [2]$

Illustrative Examples

Example 1: Let T be the set of triangle in a plane and R = {(x, y) : x, y$\in$T and x is similar to y}. Then R is (i) Reflexive (ii) Symmetric (iii) Transitive.

Solution:

(i) R is Reflexive: Let x$\in$T then clearly (x, x)$\in$R i.e xRx $\forall$ x$\in$T [Since every triangle x is similar to itself]

Hence R is a reflexive relation.

(ii) R is Synmetric : Let x, $y \in T$ such that $(x, y) \in R$ i.e xRy

Since $(x, y) \in R \Rightarrow$ x is similar to y

$\Rightarrow$ y is similar to x

$\Rightarrow (y, x) \in R$ i.e yRx

Hence we see that if $(x, y) \in R$ then $(y, x) \in R \;\forall\; x, y \in T$

So R is a symmetric relation.

(iii) R is Transitive: Let x, y, $z \in T$ such that $(x, y) \in R$ and $(y, z) \in R$ i.e xRy and yRz

Since $(x, y) \in R \Rightarrow$ x is similar to y

Also $(y, z) \in R \Rightarrow$ y is similar to z

Thus x is similar to y and y is similar to z

$\Rightarrow$ x is similar to z

$\Rightarrow (x, z) \in R$ i.e xRz

Here we see that if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R \;\forall\; x, y, z \in T$

So R is a Transitive relation.

Therefore R is Reflexive, Symmetric and Transitive relation and hence R is an equivalence relation.

Example 2. Let S be the set of straight lines in a plane and R be a relation defined in S by the rule $R = \{(x, y) : x, y \in S$ and x is perpendicular to y$\}$. Then R is Symmetric but not Reflexive and Transitive.

Solution:

(i) R is not Reflexive: Let $x \in S$. Then any straight line x cannot be perpendicular to itself i.e x is not perpendicular to x and hence $(x, x) \notin R$ i.e $x \not R x$.

Hence R is not a reflexive relation.

(ii) R is symmetric: Let x, $y \in S$ such that $(x, y) \in R$ i.e xRy

Since $(x, y) \in R \Rightarrow$ x is perpendicular to y

$\Rightarrow$ y is perpendicular to x

$\Rightarrow$ y is perpendicular to x

$\Rightarrow (y, x) \in R$ i.e yRx

Thus we see that if $(x, y) \in R$ then $(y, x) \in R \;\forall\; x, y \in S$

Hence R is a symmetric relation.

(iii) R in not Transitive: Let x, y, $z \in S$ such that $(x, y) \in R$ and $(y, z) \in R$ i.e xRy and yRz $\forall$ x, y, $z \in S$

Since, $(x, y) \in R \Rightarrow x$ is perpendicular to y

Also $(y, z) \in R \Rightarrow y$ is perpendicular to z

Hence x is perpendicular to y and y is perpendicular to z implies that x is not perpendicular to z. [infact x is parallel to z] i.e $(x, z) \notin R$

Thus we see that $(x, y) \in R$ and $(y, z) \in R \not\Rightarrow (x, z) \in R$ i.e $x \not R z$

Hence R is not a Transitive relation.

Example 3. In the set of natural numbers IN, define a relation R by the rule xRy if and only if x.y is the square of a natural number. Examine whether R is an equivalence relation or not. (NEHU, 2004, 2006)

Solution:

(i) Reflexivity: Let $x \in IN$. Then clearly $x.x = x^2$ is a square of the natural number and hence xRx.

Hence R is a reflexive relation.

(ii) Symmetricity: Let $x, y \in IN$ such that $(x, y) \in R$ i.e xRy

Since xRy then x.y is square of a natural number

i.e $xy = a^2$ for some $a \in IN$

i.e $y.x = a^2$

i.e y.x is a square of a natural number

i.e yRx

Thus we see that if xRy then yRx $\forall x, y \in IN$

Hence R is a Symmetric relation.

(iii) Transitivity: Let $x, y, z \in IN$ such that

xRy and yRz. Then since

$xRy \Rightarrow x.y$ is a square of a natural number

i.e $x.y = a^2$ for some $a \in IN$

Also $yRz \Rightarrow y.z = b^2$ for some $b \in IN$

Now $(x.y)(y.z) = a^2b^2$

$$\Rightarrow x.z = \frac{a^2b^2}{y^2} \Rightarrow x.z = \left(\frac{ab}{y}\right)^2 \in IN$$

Thus x.z is a square of a natural number and hence xRz

Thus we see that xRy and $yRz \Rightarrow xRz$ $\forall x, y \in IN$

Hence R is a Transitive relation

Since R is reflexive, Symmetric and Transitive hence R is an equivalence relation.

Example 4: In the set of integers $\mathbb{Z}$, define a relation "congruence modulo 4". Show that the relation is an equivalence relation and find all the disjoint equivalence classes into which $\mathbb{Z}$ is partitioned. (NEHU, 2005, 2007)

Solution: Let R be the relation. Then

$R = \{(a, b) : a, b \in \mathbb{Z}$ and a is congruent to b modulo 4$\}$

i.e $R = \{(a, b) : a, b \in \mathbb{Z}$ and $a-b$ is divisible by 4$\}$

We now show that R satisfies the condition of reflexivity, Symmetricity and Transitivity.

(i) R is reflexive: Let $a \in \mathbb{Z}$. Then $a-a=0$, is divisible by 4.

i.e a is conguent to a modulo 4

i.e $(a, a) \in R \ \forall\ a \in \mathbb{Z}$

Hence R is reflexive.

(ii) R is Symmetric: Let $a, b \in \mathbb{Z}$ such that $(a, b) \in R$

Since $(a, b) \in R \Rightarrow$ a is congruent to b modulo 4

$\Rightarrow a-b$ is divisible by 4

$\Rightarrow a-b = 4k$ for some $k \in \mathbb{Z}$

Now $a-b = 4k \Rightarrow -(b-a) = 4k \Rightarrow b-a = 4(-k) - k \in \mathbb{Z}$

$\Rightarrow b-a$ is divisible by 4

i.e b is congruent to a modulo 4

i.e $(b, a) \in R$

Thus we see that if $(a, b) \in R$, then $(b, a) \in R$

Hence R is symmetric

(iii) R is Transitive: Let $a, b, c \in \mathbb{Z}$ such thatg $(a, b) \in R$ and $(b, c) \in R$

Since $(a, b) \in R \Rightarrow$ a is congruent to b modulo 4

$\Rightarrow a-b$ is divisible by 4

$\Rightarrow a-b = 4m$ for some $m \in \mathbb{Z}$

Also $(b, c) \in R \Rightarrow$ b is congruent to be modulo 4

$\Rightarrow b-c$ is divisible by 4

$\Rightarrow b-c = 4n$ for some $n \in \mathbb{Z}$

Now $a-b=4m$ and $b-c=4n$

$\therefore (a-b) + (b-c) = 4m + 4n$

$\Rightarrow$ $a-c = 4(m+n) = 4k$ (say) $k \in \mathbb{Z}$

$\Rightarrow$ $a-c$ is divisible by 4

$\Rightarrow$ a is congruent to c modulo 4

$\Rightarrow$ $(a, c) \in R$

Thus we see that if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$

Hence R is Transitive

Therefore R is an equivalence Relation.

We now find all disjoint equivalence class of $\mathbb{Z}$

$\bar{0} = [0] = \{x:\in \mathbb{Z}: (x, 0) \in R\}$
$= \{x \in \mathbb{Z}: x \text{ is congruent to } 0 \text{ modulo } 4\}$
$= \{x \in \mathbb{Z}: x-0 \text{ is divisible by } 4\}$
$= \{0, \pm 4, \pm 8, \pm 12, ...\}$

$\bar{1} = [1] = \{x \in \mathbb{Z}: (x, 1) \in R\}$
$= \{x \in \mathbb{Z}: x \text{ is congruent to } 1 \text{ modulo } 4\}$
$= \{x \in \mathbb{Z}: x-1 \text{ is divisible by } 4\}$
$= \{1, -3, 5, -7, 9, -11, ...\}$

$-\bar{1} = [-1] = \{x \in \mathbb{Z}: (x, -1) \in R\}$
$= \{x \in \mathbb{Z}: x \text{ is congruent to } -1 \text{ modulo } 4\}$
$= \{x \in \mathbb{Z}: x-(-1) \text{ is divisible by } 4\}$
$= \{-1, 3, -5, 7, -9, ...\}$

$\bar{2} = [2] = \{x \in \mathbb{Z}: (x, 2) \in R\}$
$= \{x \in \mathbb{Z}: x \text{ is congruent to } 2 \text{ modulo } 4\}$
$= \{x \in \mathbb{Z}: x-2 \text{ is divisible by } 4\}$
$= \{\pm 2, \pm 6, \pm 10, \pm 14, ...\}$

$-\bar{2} = [-2] = \{\pm 2, \pm 6, \pm 10, \pm 14, ...\}$

$\bar{3} = [3] = \{-1, 3, -5, 7, -9, ...\}$

$-\bar{3} = [-3] = \{1, -3, 5, -7, ...\}$

$\bar{4} = [4] = \{0, \pm 4, \pm 8, \pm 12, ...\}$

$-\bar{4} = [-4] = \{0, \pm 4, \pm 8, \pm 12, ...\}$

Thus we see that

$[0] = [4] = [8] = ... = [-4] = [-8] = ...$

$[1] = [5] = [9] = ... = [-3] = [-7] = ...$

$[2] = [6] = [10] = ... = [-2] = [-6] = ...$

$[3] = [7] = [11] = ... = [-1] = [-5] = ...$

Thus [0], [1], [2], [3] are the only disjoint equivalence class into which $\mathbb{Z}$ is partitioned

i.e $\mathbb{Z} = [0] \cup [1] \cup [2] \cup [3]$

Example 5: Let IN be the set of natural number and R be a relation on IN defined by xRy if and only if x divides y $\forall$ x,y$\in$IN. Examine whether R is reflexive, symmetric, antisymmetric and transitive. (NEHU, 2012)

Solution: R = {xRy : x, y$\in$IN and x divides y}

(i) Reflexivity: Let x$\in$IN. Then x always divides x (every natural number divides itself)

Hence xRx $\forall$ x$\in$IN

So R is reflexive

(ii) Symmetricity: Let x, y$\in$IN such that xRy

Since xRy $\Rightarrow$ x divides y

But x divides y does not necessarily imply that y also divides x (2 divides 4 but 4 does not divide 2)

Hence xRy $\not\Rightarrow$ yRx

So R is not Symmetric

(iv) Transitivity: Let x, y, z$\in$IN such t hat xRy and yRz

Since xRy $\Rightarrow$ x divides y $\Rightarrow$ y = ax for some a$\in$IN

Also yRz $\Rightarrow$ y divides z $\Rightarrow$ z = by for some b$\in$IN

Now z = by = b (ax) = (ab)x = cx where c=ab$\in$IN

i.e x divides z $\Rightarrow$ xRz

Thus we see that if xRy and yRz then xRz $\forall$ x, y, z$\in$IN

So R is transtive.

Example 6. Let R be an equivalence relation an as set x. Then

(i) $[x] = [y] \Leftrightarrow (x, y) \in R$

(ii) $[x] \neq [y] \Leftrightarrow [x] \cap [y] = \phi$

Where [x] = equivalence class of x (NEHU 2001, 2008, 2014, 2016)

Solution: $[x] = \{a \in x:\ xRa \text{ i.e } (a, x) \in R\}$

Suppose $[x] = [y]$

Then $\exists$ $a \in x$ such that $(a, x) \in R$

Since $[x] = [y] \Rightarrow (a, y) \in R$

Now since R is an equivalence relation, it is reflexive

Hence $(a, x) \in R \Rightarrow (x, a) \in R$

Also since R is an equivalence relation, it is tramsitive.

Therefore $(x, a) \in R$ and $(a, y) \in R \Rightarrow (x, y) \in R$

Conversely suppose $(x, y) \in R$. Then since R is equivalence relation $\exists$ $a \in x$ such that $(x, a) \in R$ and $(a, y) \in R \Rightarrow [x] = [y]$

(ii) Suppose $[x] \neq [y]$

We prove that $[x] \cap [y] = \phi$

Suppose $a \in [x] \cap [y] \Rightarrow a \in [x]$ and $a \in [y]$

Since $a \in [x] \Rightarrow (x, a) \in R$

Also $a \in [y] \Rightarrow (y, a) \in R$

But $[x] \neq [y]$ hence $(x, a) \in R$ and $(y, a) \in R$ is not possible and therefore $[x] \cap [y] = \phi$

Example 7. Let R and R^1 be two equivalence relations on a set A. Then $R \cap R^1$ is an equivalence relation on A. (NEHU, 2007)

Solution: Since R and R^1 are equivalence relation on A

$\therefore$ $R \subseteq A \times A$ and $R^1 \subseteq A \times A$

Hence $R \cap R^1 \subseteq A \times A$

So $R \cap R^1$ is a relation on A

(i) Reflexivity:

Since R and R^1 are equivalence relation on A

$\therefore$ R and R^1 are reflexive

Hence $(a, a) \in R$ and $(a, a) \in R^1$ $\forall$ $a \in A$

i.e $(a, a) \in R \cap R^1$ $\forall$ $a \in A$

Hence $R \cap R^1$ is also reflexive

(ii) Symmetricity:

Let $a, b \in A$ such that $(a, b) \in R \cap R^1$

Then $(a, b) \in R$ and $(a, b) \in R^1$

Since R and R^1 are equivalence relation,

$\therefore$ R and R^1 are symmetric

Hence $(a, b) \in R \Rightarrow (b, a) \in R$ and $(a, b) \in R^1 \Rightarrow (b, a) \in R^1$

Hence $(b, a)\in R$ and $(b, a)\in R^I \Rightarrow (b, a)\in R\cap R^I$

Hence $(a, b)\in R\cap R^I \Rightarrow (b, a)\in R\cap R^I \,\forall\, a, b\in A$

Therefore $R\cap R^I$ is also symmetric

(iii) Transitivity:

Let $a, b, c\in A$ such that $(a, b)\in R\cap R^I$ and $(b, c)\in R\cap R^I$

Since $(a, b)\in R\cap R^I \Rightarrow (a, b)\in R$ and $(a, b)\in R^I$

Also $(b, c)\in R\cap R^I \Rightarrow (b, c)\in R$ and $(b, c)\in R^I$

Hence $(a, b)\in R$ and $(b, c)\in R$ and $(a, b)\in R^I$ and $(b, c)\in R^I$

Since R and R^I are both transitive relations

Therefore $(a, b)\in R$ and $(b, c)\in R \Rightarrow (a, c)\in R$

Also $(a, b)\in R^I$ and $(b, c)\in R^I \Rightarrow (a, c)\in R^I$

Hence $(a, c)\in R$ and $(a, c)\in R^I \Rightarrow (a, c)\in R\cap R^I$

Thus $(a, b)\in R\cap R^I$ and $(b, c)\in R\cap R^I \Rightarrow (a, c)\in R\cap R^I$

Hence $R\cap R^I$ is transitive

Since $R\cap R^I$ is Reflexive, Symmetric and Transitive

$\therefore$ $R\cap R^I$ is an equivalence relation on A.

Example 8. Let A be a non empty set and $\sim$ be an equivalence relation on A. For an arbitrary element $a\in A$, define $\bar{a} = \{x\in A : x\sim a\}$. Show that (i) $a\in \bar{a}$ and (ii) if $b\in \bar{a}$ then $\bar{b} = \bar{a}$ where $b\in A$. (NEHU 2002)

Solution: (i) We have $\bar{a} = \{x\in A : x\sim a\}$

Since $\sim$ is an equivalence relation, $\sim$ is reflexive

Hence for any $a\in A$, $a \sim a$ i.e $a\in \bar{a}$

(ii) If $b=\bar{a}$ then $b \sim a$

Let $x\in \bar{b}$, $x\in A$, Then $x\sim b$

But $b\sim a$ and $\sim$ is transitive

$\therefore$ $x\sim b$ and $b \sim a \Rightarrow x\sim a \Rightarrow x\in \bar{a}$

i.e $\bar{b}\subseteq \bar{a}$

Conversely if $y\in \bar{a}$, $y\in A$. Then $y\sim a$

Now $b\sim a \Rightarrow a\sim b$ since $\sim$ is symmetric

Hence $y\sim a$ and $a\sim b \Rightarrow y\sim b \Rightarrow y\in \bar{b}$

i.e $\bar{a}\subseteq \bar{b}$

Hence $\bar{a}=\bar{b}$

Exercise 2.1

1. Find x and y if (x+2, 4) = (5, 2x+y) (Ans: x = 3,y = –2)

2. Let A = {1, 2, 3, 4} and S = {(a, b) : a∈A, b∈A, a divides b}. Write S explicitly. (Ans: S= {(1, 1), (1, 2), (1, 3), (1, 4),
(2, 2), (2, 4), (3, 3), (4, 4)}

3. Let A and B be two subsets such that n(A) = 3 and n (B) = 2. If (x, 1), (y, 2), (z, 1) are in A×B, find A and B, where x, y, z are district elements. (Ans: A = {x, y, z}, B = (1, 2})

4. If A = {1, 2, 3}, B = {3, 4} and C = {4, 5, 6}. Find:

   (i) A×(B∩C) (ii) (A×B) ∩ (A×C) (iii) A× (B∪C)

   (iv) (A× B) ∪ (A×C)

   (Ans: (i) {(1, 4), (2, 4), (3, 4)}, (ii) {(1, 4), (2, 4), (3, 4)}

   (iii) {(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3) (3, 4), (3, 5), (3, 6)}

   (iv) Same as (iii))

5. If A and B be non empty subsets, the show that A×B = B×A if and only if A=B

6. Let A be a non empty set such that A×B = A×C. Show that B=C.

7. For any three sets A, B and C, prove that

   (i) (A–B)×C = (A×C) – (B×C)

   (ii) A×(B∪C) = (A×B) ∪ (A×C)

   (iii) (A∩B)×C = (A×C) ∩ (B×C)

8. Let A = {1, 2, 3, 4, 6}. Let R be the relation on A defined by:

   {(a, b) : a∈A, b∈A, a divides b}

   Find (i) R (ii) domain of R (iii) Range of R

9. Let A = {3, 5} and B = {7, 11} Let

   R = {(a, b) : a∈A, b∈B, a–b is odd}

   Show that R is an empty relation from A to B.

10. Let R be the relation on Z defined by aRb if and only if a–b is an even interger. Find:

    (i) R (ii) domain of R (iii) range of R

    Is R an equivalence relation? Justify.

11. Let R be the relation on Z defined by:

    $R = \{(a, b) : a, b \in Z \text{ and } a^2 = b^2\}$

    Find: (i) R (ii) domain of R (iii) range or R

    Is R an equivalence relation?

12. Give an example of a relation, which is –

    (i) reflexive but neither symmetric nor transitive

    (ii) reflexive, symmetric but not transitive

    (iii) symmetric and transitive but not reflexive

    (iv) reflexive and anti symmetric (NEHU, 2015)

Ans: (i) (ii) $R=\{(a, b): a, b \in R \text{ and } 1 + ab > 0\}$

(iii) $R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}$; $A = \{1, 2, 3\}$

13. Define a relation R on the set of integers Z as follows:

    aRb if and only if "a–b is divisible by 5"

    Show that R is an equivalence relation on Z.

    Describe all equivalence classes and hence show that R gives rise to a partition on Z. (NEHU, 2011, 2013)

14. Let a relation R in the set of natural numbers IN be define by $xRy \Leftrightarrow (x-y)(x-3y) = 0$. Determine whether R is an equivalence relation? (NEHU, 2013)

15. A relation R on the set IN×IN is defined by (a, b) R (c, d) if and only if $a+d = b+c$. Show that R is an equivalence relation. Also find the equivalence class of $(2, 3) \in IN \times IN$. (NEHU, 2006)

16. Give an example to show that the union of two equivalence relations an a set need not be an equivalence relation.

    (Ans: Hint $A = \{(1, 2, 3)$ $R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1)\}$

    $R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3), (3, 1)\}$)

17. Let IN be the set of natural number and let R be a relation in IN defined by

    $R = \{(a, b) : a, b \in IN \text{ and a is multiple of b}\}$

    Show that R is reflexive, and transitive but not symmetric.

18. Let L be the set of all lines in a plane and let R be a relation on L defined by $R = \{(l_1, l_2) : l_1, l_2 \in L$ an l_1 is parrellel to $l_2\}$. Determine whether R is an equivalence relation.

19. On the set of real numbers R define a relation R by R = {(a, b) : a, b∈R and 1+ab>0? Show that R is reflexive and symmetric but not transitive.

20. Let R be the set of real numbers and R be a relation on IR define by R= {(a, b) : a, b∈IR and $a \leq b^2$}. Show that R is satisfies none of reflexivity, symmetry and transitivity.

Functions:

2.9 Definition:

Let A and B be two non empty sets. A relation 'f' from A to B which associates every element x of A to a unique elemennt y of B is called a 'function' or a 'mapping' or a 'coresspondence' or 'transformation'.

If f is a function from A to B, it is denoted by $f:A \to B$.

The unique element y of B is called the 'value' of f at x or the "image" of x under f and is written as y=f(x).

The element x of A is called the pre-image (or inverse image) of y.

The set A is called the domain of f and the set B is called the co-domain of f.

The set of all images of the elements of A under f is called the range of f and is denoted by Range (f) or f(A).

Thus Range (f) = {f(x) : for all x∈A}

Note: The Range of f is a subset of B which may or may not be equal to B.

Remarks:

(i) To each element x in A, there exists a unique element y of B such that y=f(x)

(ii) Different elements of A may be associated with the same element of B.

(iii) There may exists some element of B which are not associated with any element of A.

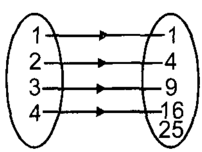
Example:

1. Let A = {1, 2, 3, 4} and B = {1, 4, 9, 16, 25}

Consider the rule $f:A \to B : f(x) = x^2 \ \forall\ x \in A$

Then each element in A has a unique image in B.

So f is a function from A to B

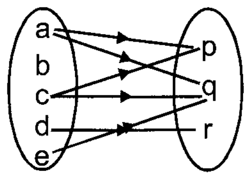


Dom (f) = {1, 2, 3, 4} = A; co-domain (f) = {1, 4, 9, 16, 25} = B

Range (f) = {1, 4, 9, 16}

Clearly $25 \in B$ but it does not have pre-image in A.

2. Let A = {a, b, c, d, e} and B = {p, q, r}. The rule depicted in the following diagram which associates the elements a and c in A with two elements p and q of B, is therefore not a function from A to B.



3. Let IN be the set of natural numbers and consider the rule $f: \mathbb{N} \to \mathbb{N}$: $f(x) = 2x, \forall x \in \mathbb{N}$

Here clearly every element x in IN has a unique image 2x in IN and hence f is a function from IN to IN.

Clearly f(1) = 2, f(2) = 4, f(3) = 6,...... and so on.

Dom (f) = IN; co-domain (f) = IN Range (f) = {2, 4, 6,...}

2.10 Types of Functions

(a) One-one Function or Injective Function

A function $f: A \to B$ is said to be one-one (or injective) if and only if distinct element of A have distinct images in B

i.e. iff $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ or iff $x_1 = x_2 \Rightarrow f(x_1) = f(x_2) \ \forall x_1, x_2 \in A$

A function which is not one-one is called many one.

Example:

Let IN be the set of natural numbers

Let $f: \mathbb{N} \to \mathbb{N} : f(x) = 2x, \forall x \in \mathbb{N}$

Then $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2 \ \forall x_1, x_2 \in \mathbb{N}$

Hence f is a one-one function.

(b) Onto Function or Surjective Function:

A function $f: A \to B$ is said to be onto (or surjective) if and only if every element in B is an image of at least one element in A.

Thus f is onto iff for each $y \in B$, $\exists$ at least one element $x \in A$ such that y = f(x)

Also f is onto$\Leftrightarrow$range (f) = B.

A function which is not onto is called into.

Example:

Let IR be the set of real numbers

Let $f:IR \to IR$: $f(x) = 3x-2 \ \forall \ x \in IR$

Then if $y \in IR$ such that y = f(x) i.e y=3x–2, then $x = \frac{y+2}{3} \in IR$

Thus for each $y \in IR$ there exist to $x = \frac{y+2}{3} \in IR$

such that $f(x) = f\left(\frac{y+2}{3}\right) = \cancel{3}\left(\frac{y+2}{\cancel{3}}\right) - 2 = y$

This shows that every element in the co-domain has a pre-image in the domain.

Hence f is an onto function.

(c) Bijective Function or one-to-one Correspondence:

A one-one and onto function is said tobe a bijective function or a one-to-one correspondence.

(d) Constant Function

A function $f:A \to B$ is called a constant function if every element of A has the same image in B

In other words, a function $f:A \to B$ is called a constant function if f(x) = k, $\forall \ x \in A$ and $k \in B$

Example:

Let A = {1, 2, 3, 4} and B = {5, 6, 7}

Let $f:A \to B$: f(x) = 5 for all $x \in A$

Then clearly every element in A has same image in B.

So f is a constant.

(e) Identity Function:

The function $f:A \to B$ is called an identity function if $f(x) = x \ \forall \ x \in A$

For the Identity Function Domain (f) = Range (f) = A

(f) Equal Functions:

Two functions f and g having same domain D are said to be equal if f(x) = g(x) $\forall \ x \in D$

Example: $f(x) = \frac{x^2-1}{x+1}$ and g(x) = x–1 are equal functions

(g) Even and odd Function:

A function f defined on a certain domain D is said to be an even function if $f(-x) = f(x) \quad \forall\ x \in D$

It is said to be an odd function if $f(-x) = -f(x)\ \forall\ x \in D$

Example

(i) On the set of integers Z Let $f: Z \rightarrow Z$ be such that $f(x) = x^2 \quad \forall\ x \in Z$

Then clearly $f(-x) = (-x)^2 = x^2 = f(x) \quad \forall\ x \in Z$

Hence f is an even function.

(ii) On the set of real numbers IR, let $f: IR \rightarrow IR$ be such that f(x) = sinx $\forall\ x \in IR$

Then clearly $f(-x) = \sin(-x) = -\sin x = -f(x) \quad \forall\ x \in R$

Hence f is an odd function.

(h) Periodic Function

A function f defined on a certain domain D is said to be a periodic function with period α if

$f(x + \alpha) = f(x)\ \forall\ x \in D$

where α is least positive real number/constant.

Example:

Let $f: IR \rightarrow IR$ be such that f(x) = sinx

Then $f(2\pi+x) = \text{Sin}(2\pi+x) = \text{Sin}x = f(x)$

Hence f(x) = Sinx is a periodic function with period 2π.'

Similarly, cosx, secx and cosecx are periodic functions with period 2π and tanx, cotx are periodic functions with period π.

2.11 Algebra of Functions

Like real numbers, the algebraic operations of addition, subtraction, multiplication and division (zero function is excluded in case of division) yield new functions.

Let f and g be two real valued functions defined in certain domains D' and D'' respectively and let $D = D' \cap D'' \neq \phi$.Then

(i) The sum function denoted by f + g, is defined by (f+g) (x) = f(x) + g(x) with domain D

(ii) The Difference function denoted by f–g, is defined by $(f-g)(x) = f(x) - g(x)$ with domain D

(iii) The Product function denoted by fg, is defined by $(fg)(x) = f(x).g(x)$ with domain D

(iv) The Quotient function denoted by $\frac{f}{g}$, is defined by $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ with domain $\{x : x \in D, g(x) \neq 0\} \neq \phi$

Remark:

(a) If f is a function, then ff is denoted by f^2, f^2f is denoted by f^3 and so on where $(f^n)(x) = (f(x))^n$ $n \in IN$, $x \in D'$

(b) The receprocal function $\frac{1}{f}$, is defined by $\left(\frac{1}{f}\right)(x) = \frac{1}{f(x)}$ with domain $\{x : x \in D;\ f(x) \neq 0\}$

(c) If 'k' is any real number, then scalar multiple of f by k denoted by kf is defined by $(kf)(x) = kf(x)$ with domain D'.

2.12 Composition of Functions

Let $f: A \to B$ and $g: B \to C$ be two given functions then the composition of f and g, denoted by gof is the function defined by

$(gof): A \to C : (gof)(x) = g(f(x))\ \forall\ x \in A$

Clearly Dom (gof) = dom (f)

Also gof is defined only if Range (f) $\subseteq$ Dom (g)

Example 1:

Let $f: \{1, 3, 4\} \to \{1, 2, 5\}$ and $g: \{1, 2, 5\} \to \{1, 3\}$ be defined as $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(1, 3), (2, 3), (5, 1)\}$. Find (fog) and (gof)

Solution:

Clearly Range (g) = $\{1, 3\} \subseteq$ Dom (f) $= \{1, 3, 4\}$

$\therefore$ fog is defined and dom (fog) = dom (g) = $\{1, 2, 5\}$

Now $(fog)(1) = f(g(1)) = f(3) = 5$

$(fog)(2) = f(g(2)) = f(3) = 5$

$(fog)(5) = f(g(5)) = f(1) = 2$

Hence fog = $\{(1, 5), (2, 5), (5, 2)\}$

Also Range f = $\{1, 2, 5\} \subseteq$ Dom (g) = $\{1, 2, 5\}$

$\therefore$ gof is defined and dom(gof) = dom(f) = $\{1, 3, 4\}$

Now $(gof)(1) = g(f(1)) = g(2) = 3$

(gof) (3) = g(f(3)) = g(5) = 1

(gof) (4) = g(f(4)) = g(1) = 3

Hence (gof) = {(1, 3), (3, 1), (4, 3)}

Example 2:

Let $f:IR \rightarrow IR : f(x) = 8x^3$ and $g:IR \rightarrow IR : g(x) = x^{\frac{1}{3}}$. Find (gof) and (fog)

Solution: Let $x \in IR$, then

$$(gof)(x) = g(f(x)) = g(8x^3) = (8x^3)^{\frac{1}{3}} = 2x$$

$$(fog)(x) = f(g(x)) = f(x^{\frac{1}{3}}) = 8\left(x^{1/3}\right)^3 = 8x$$

$\therefore$ gof $\neq$ fog.

2.13 Invertible Function

Let $f:A \rightarrow B$ be a function. If there exists another function $g:B \rightarrow A$ such that $gof = I_A$ and $fog = I_B$ then f is called an invertible function and g is called the inverse of f denoted by f^{-1}

Remark: $fof^{-1} = I_B$ and $f^{-1}of = I_A$

Example: Let $f:IR \rightarrow IR$ be such that $f(x) = x + 2$

Let $y = f(x) \Rightarrow y = x + 2 \Rightarrow x = y - 2 \Rightarrow f^{-1}(y) = y-2$

Hence we define $f^{-1} : IR \rightarrow R: f^{-1}(y) = y-2$

Theorem 1: Let $f:A \rightarrow B$ be a function. If f is one-one and onto, then f is invertible i.e f^{-1} exists.

Proof: Since $f:A \rightarrow B$ is one-one and onto, then there exists a unique $x \in A$ such that for any $y \in B$, $y=f(x)$

Consider a function $g:B \rightarrow A$ such that $g(y) = x$. Then

$(gof)(x) = g(f(x)) = g(y) = x = I_A(x)$

$\therefore gof = I_A$

Also $(fog)(y) = f(g(y)) = f(x) = y = I_B y$

$\therefore fog = I_B$

Hence, f is invertible and $f^{-1}=g$.

Theorem 2: If $f:A \rightarrow B$ is an invertible function, then f is one-one and onto

Proof: Since $f:A \rightarrow B$ is an invertible function, then there exists another functions $g:B \rightarrow A$ such that $gof=I_A$ and $fog=I_B$

Let $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$

Then $g(f(x_1)) = g(f(x_2))$

$\Rightarrow (gof)(x_1) = (gof)(x_2)$

$\Rightarrow I_A(x_1) = I_A(x_2)$

$\Rightarrow x_1 = x_2$

Hence f is one-one

Next, Let $y \in B$. Since $g:B \to A$ is a function, hence f a unique $x \in A$ such that $g(y) = x$

$\therefore \quad f(g(y)) = f(x)$

$\Rightarrow \quad (fog)(y) = f(x)$

$\Rightarrow \quad I_B(y) = f(x)$

$\Rightarrow \quad y = f(x)$

Hence for each $y \in B$, $\exists x \in A$ such that $y = f(x)$

$\therefore$ f is onto.

Hence f is one-one and onto.

Theorem 3: A function f is invertible if and only if it is 9 bijective function.

or

f^{-1} exists if and only if f is both one-one and onto.

Proof: Follows from theorem 1 and theorem 2.

Theorem 4: Invertible function has a unique inverse

or

Inverse of a function if exists is unique.

Proof: If possible suppose the function $f:A \to B$ has two inverses g and h

Then $fog = I_B$ and $foh = I_B$

Hence $(fog)(y) = (foh)\ y$

$\Rightarrow f(g(y)) = f(h(y)) \quad \forall\ y \in B$

$\Rightarrow g(y) = h(y) \qquad \because$ f is one-one

$\Rightarrow g = h$

Hence f has a unique inverse.

Theorem 5: Let $f:A \to B$ and $g:B \to C$ be one-one and onto functions. Then $gof:A \to C$ is also one-one and onto and $(gof)^{-1} = f^{-1}og^{-1}$ (NEHU, 2013)

Proof: We first show that gof is one-one

Let $x_1, x_2 \in A$ such that

$(gof)(x_1) = (gof)(x_2)$

$\Rightarrow g(f(x_1)) = g(f(x_2))$

$\Rightarrow f(x_1) = f(x_2)$ since $g:B \to C$ is one-one

$\Rightarrow x_1 = x_2$ since $f:A \to B$ is one-one

Hence $(gof)(x_1) = (gof)(x_2) \Rightarrow x_1 = x_2 \ \forall\ x_1, x_2 \in B$

$\therefore$ gof is one-one

We now prove that gof is onto

Let $z \in C$, then since $g:B \to C$ is onto, hence to each $z \in C$ there exist $y \in B$ such that $g(y) = z$.

Also since $f:A \to B$ is onto, hence to each $y \in B$, there exists $x \in A$ such that $f(x) = y$

Now $z = g(y) = g(f(x)) = (gof)(x)$

Thus for each $z \in C$, there exists $x \in A$ such that $(gof)(x) = z$

Hence gof is onto.

$\therefore$ gof is both one-one and onto and hence $(gof)^{-1}$ exists.

Since $f:A \to B$ and $g:B \to C$ are invertible functions

$\therefore y = f(x)$ and $z = g(y)$ $x = f^{-1}(y)$ and $y = g^{-1}(z)$

Also $(gof)(x) = z \Rightarrow (gof)^{-1}(z) = x$

Again $(f^{-1}og^{-1})(z) = f^{-1}(g^{-1}(z)) = f^{-1}(y) = x$

$\therefore (gof^{-1})(z) = (f^{-1}og^{-1})(z)$

Hence $(gof)^{-1} = f^{-1}og^{-1}$

Illustrative Examples

Example 1: Let $f:IN \to IN$ be such that $f(x) = 2x \ \forall \in IN$. Show that f is one-one and into i.e not onto.

Solution: Let $x_1, x_2 \in IN$ such that $f(x_1) = f(x_2)$

Then $2x_1 = 2x_2 \Rightarrow x_1 = x_2$

$\therefore$ f is one one.

Let $y = 2x \in IN \Rightarrow x = \dfrac{y}{2}$

If $y = 3$, then $x = \dfrac{3}{2} \notin IN$

Thus $3 \in IN$ has no pre image in IN under f

$\therefore$ f is not onto and hence f is into

$\therefore$ f is one-one and into

Example 2: Show that the function f:IR→IR: $f(x) = x^2$ is neither one-one nor onto.

Solution: We see that $f(1) = 1^2 = 1$ and $f(-1) = (-1)^2 = 1$

$\therefore$ $f(1) = f(-1)$ but $1 \neq -1$

Hence f is not one-one

Also $-1 \in IR$ has no pre image in IR under f

$\therefore$ f is also no onto

Hence f is neither one-one nor onto

Example 3: Find the domain and range of the real function $f(x) = \sqrt{9-x^2}$

Solution: Clearly $f(x) = \sqrt{9-x^2}$ is not defined if $9-x^2 < 0$

i.e when $9-x^2 < 0$ i.e $(x-3)(x+3) > 0$

i.e $x > 3$ or $x < -3$

$\therefore$ Dom (f) = $\{x \in IR : -3 \leq x \leq 3\} = [-3, 3]$

Also if $y = \sqrt{9-x^2} \Rightarrow y^2 = 9-x^2 \Rightarrow x = \sqrt{9-y^2}$

Clearly x is not defined if $9-y^2 < 0$

i.e when $y^2 > 9$ i.e $(y-3)(y+3) > 0$

i.e $y>3$ or $y<-3$

$\therefore$ Range (f) = $\{yIR : -3 \leq y \leq 3\} = [-3, 3]$

Example 4: Find the domain and range of the real function

$$f(x)= \frac{1}{1-x^2}$$

Solution: Clearly $f(x) = \frac{1}{1-x^2}$ is not defined if $1-x^2=0$

i.e when $x^2 = 1$ i.e $x = \pm 1$

$\therefore$ Dom(f) = IR – [–1, 1]

If $y = \frac{1}{1-x^2} \Rightarrow 1-x^2 = \frac{1}{y} \Rightarrow 1-\frac{1}{y} = x^2 \Rightarrow x = \sqrt{1-\frac{1}{y}}$

Clearly x is not defined if $1-\frac{1}{y} < 0$ i.e $1<\frac{1}{y}$ i.e $y<1$

Range (f) $R = IR - \{y \in IR : y<1\} = \{y \in IR : y \geq 1\}$

Example 5: Let $f:A \rightarrow B$ and I_A and I_B be identity functions and A and B respectively. Prove that $(foI_A) = f$ and $(I_Bof) = f$.

Solution: Let $x \in A$ and $y=f(x)$ then

$(foI_A)(x) = f(I_A(x)) = f(x)$

$\therefore (foI_A) = f$

Also $(I_Bof)(x) = I_B(f(x)) = I_B(y) = y = f(x)$

$\therefore (I_Bof) = f$

Hence $(foI_A) = f$ and $(I_Bof) = f$

Example 6: Let $f:A \rightarrow B$, $g:B \rightarrow C$ and $h:C \rightarrow D$. Then prove that $(hog)of = ho(gof)$ (Associative Law)

Solution: Since $f:A \rightarrow B$, for each $x \in A$ $\exists$ a unique $y \in B$ such that $y=f(x)$

Also $g:B \rightarrow C$, for each $y \in B$, $\exists$ a unique $z \in C$ such that $z=g(y)$

Again $h:C \rightarrow D$, for each $z \in C$ $\exists$ a unique $w \in D$ such that $w=h(z)$

Now $[(hog)of](x) = (hog)(f(x)) = (hog)(y)$

$= h(g(y)) = h(z) = w$

Also $[ho(gof)](x) = h[(gof)(x)] = h[g(f(x))]$

$= h[g(y)] = h(z) = w$

$(hog)of = ho(gof)$

Example 7: Let $A:A \rightarrow B$ and $g:B \rightarrow A$ such that $(gof) = I_A$ show that f is $f:A \rightarrow B$ one-one and g is onto

Solution: Let $x_1, x_2 \in A$ such that $f(x_1) = f(x_2)$

Then $g(f(x_1) = g(f(x_2))$

$\Rightarrow \quad (gof)(x_1) = (gof)(x_2)$

$\Rightarrow \quad I_A(x_1) = I_A(x_2)$

$\Rightarrow \quad x_1 = x_2$

Hence $f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \; \forall \; x_1, x_2 \in A$

$\therefore$ f is one-one

We now show that g is onto

Let $x \in A$ and $y \in B$ such that $y=f(x)$

Then $g(y) = g(f(x)) = (gof)(x) = I_A(x) = x$

Thus for each $y \in B$, $\exists\, x \in A$ such that $g(y) = x$

Hence g is onto.

Example 8: Find the inverse function of the function $f:IR_0 \to IR_0$ defined by $f(x) = \frac{1}{x}\ \forall\ x \in IR_0 = IR - \{0\}$ (NEHU, 2010)

Solution: Let $y = f(x)$. Then

$$y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$$

$$\Rightarrow f^{-1}(y) = \frac{1}{y}\ [\because y=f(x) \Rightarrow x = f^{-1}(y)]$$

Thus we define $f:IR_0 \to IR_0$ such that $f^{-1}(y) = \frac{1}{y}$.

Examle 9: Show that the mapping $f:Q \to Q$ defined by $f(x) = 2x + 3$ is one-to-one and onto, where Q is the set of rational numbers. Also find the formula that defines the inverse function f^{-1}. (NEHU, 2013)

Solution: Let $x_1, x_2 \in Q$ such that $f(x_1) = f(x_2)$

Then $2x_1 + 3 = 2x_2 + 3$

$\Rightarrow\ 2x_1 = 2x_2$

$\Rightarrow\ x_1 = x_2$

Hence $f(x_1) = f(x_2) \Rightarrow x_1 = x_2\ \forall\ x_1, x_2 \in Q$

$\therefore$ f is one-one

Let $y = f(x)$ then $y = 2x + 3$

$$\Rightarrow x = \frac{y-3}{2}$$

$$\Rightarrow f^{-1}(y) = \frac{y-3}{2}$$

[Since $y = f(x) \Rightarrow x = f^{-1}(y)$]

Thus we define $f^{-1}:Q \to Q$ such that $f^{-1}(y) = \frac{y-3}{2}$

Example 10: Find the domain of the function $f(x) = \frac{1}{\sqrt{(3-x)(x-5)}}$.

(NEHU, 2016)

Solution: $f(x) = \frac{1}{\sqrt{(3-x)(x-5)}}$ is define if $(3-x)(x-5) > 0$

i.e either $3-x > 0$ and $x-5 > 0$

or $3-x < 0$ and $x-5 < 0$

i.e either $3 > x$ and $x > 5$

or $3 < x$ and $x < 5$

$\therefore$ Dom $f(x) = \{x \in IR : 3 < x < 5\} = (3, 5)$

Example 11: Let $f(x) = \dfrac{x-1}{x+1}$, show that $\dfrac{f(x)-f(y)}{1+f(x)f(y)} = \dfrac{x-y}{1+xy}$ (NEHU, 2013)

Solution: $\dfrac{f(x)-f(y)}{1+f(x)f(y)} = \dfrac{\dfrac{x-1}{x+1} - \dfrac{y-1}{y+1}}{1+\dfrac{x-1}{x+1}\cdot\dfrac{y-1}{y+1}}$

$$= \frac{(x-1)(y+1)-(y-1)(x+1)}{(x+1)(y+1)+(x-1)(y-1)}$$

$$= \frac{xy+x-y-1-xy-y+x+1}{xy+x+y+1+xy-x-y+1}$$

$$= \frac{2(x-y)}{2(1+xy)} = \frac{x-y}{1+xy}$$

Example 12: Two function f(x) and g(x) are given as follows f(x) = log sinx; g(x) = log cosx. Show that f(x) + g(x) + log2 = f(2x)(NEHU, 2010, 2016)

Solution: Given f(x) = log sinx; g(x) = log cosx

f(x) + g(x) + log2 = logsinx + logcosx + log2

= log (2sinx cosx)

[$\because$ loga + logb + logc = logabc]

= log (sin2x)

= f(2x) $\because$ f(x) = logsinx

Hence shown

Example 13: Find the domain of the following functions on the real line (NEHU, 2006)

(i) $f(x) = \sqrt{x^2}$ (ii) $g(x) = \left(\sqrt{x}\right)^2$

Solution: (i) $f(x) = \sqrt{x^2} = x$ is defined for all real values of x

Hence Dom f(x) = IR

(ii) $g(x) = \left(\sqrt{x}\right)^2 = x$ is defined for all real values of x

Hence Dom g(x) = IR

Example 14: Find the domain of the function $f(x) = \dfrac{1}{\sqrt{|x|-x}}$ (NEHU, 2008)

Solution: f(x) is defined if

$|x| - x > 0$

i.e $|x| > x$\

i.e $x < 0$

Hence Dom f(x) = $IR^{<0}$

Example 15: Let A = IR – {3}, B = IR – {1} and let f:A→B defined by $f(x) = \dfrac{x-2}{x-3}$. Is f bijective? Give reasons. (NEHU, 2013)

Solution: f is one-one since for any $x_1, x_2 \in A$

$$f(x_1) = f(x_2) \Rightarrow \frac{x_1-2}{x_1-3} = \frac{x_2-2}{x_2-3}$$

$$\Rightarrow (x_1-2)(x_2-3) = (x_2-2)(x_1-3)$$

$$\Rightarrow x_1x_2 - 3x_1 - 2x_1 + 6 = x_1x_2 - 3x_1 - 2x_1 + 6$$

$$\Rightarrow x_1 = x_2$$

∴ f is one-one

Also let $y \in B$ such that $y = \dfrac{x-2}{x-3}$

Then $(x-3)y = x-2 \Rightarrow x = \dfrac{3y-2}{y-1}$

Clearly x is defined if $y \neq 1$

Also x = 3 will give 1=0 which is false

$$\therefore x \neq 3 \text{ and } f(x) = \frac{\frac{3y-2}{y-1}-2}{\frac{3y-2}{y-1}-3} = \frac{\frac{3y-2-2y-2}{y-1}}{\frac{3y-2-3y+3}{y-1}} = y$$

Thus for each $y \in B$, $\exists$ $x \in A$ such that $y = f(x)$

$\therefore$ f is onto

Hence f is one-one and onto and therefore bijective

Example 16: If $f:A \to B$ and $g:B \to C$ be two function then $gof:A \to C$ is one one and $f:A \to B$ is onto $\Rightarrow$ $g:B \Rightarrow C$ is one-one. Prove it. (NEHU, 2012)

Solution: Since $gof:A \to C$ and $g:B \to C$ is one one. Then

$\forall$ $x_1, x_2 \in A$ $(gof)(x_1) = (gof)(x_2) \Leftrightarrow x_1 = x_2$ ....(i)

Also since $f:A \to B$ is onto, then $\forall$ $y \in B$ $\exists$

some $x \in A$ such that $f(x) = y$ ....(ii)

Let $y_1, y_2 \in B$ such that $g(y_1) = g(y_2)$ then

$g(f(x_1)) = g(f(x_2))$ since $y_1 = f(x_1)$ and $y_2 = f(x_2)$ by (ii)

$\Rightarrow (gof)(x_1) = (gof)(x_2)$

$\Rightarrow x_1 = x_2$ [$\because$ gof is one-one]

$\Rightarrow f(x_1) = f(x_2)$ [$\because f:A \to B$ is a function]

$\Rightarrow y_1 = y_2$ [$\because$ f is onto]

$\therefore$ Thus for every $y_1, y_2 \in B$ such that $g(y_1) = g(y_2) \Rightarrow y_1 = y_2$

Hence g is one-one

Example 17: Let $f:X \to Y$ and $g:Y \to Z$ be two mapping. Prove that

(i) if gof is onto, then g is onto

(ii) if gof is one-to-one, then f is one-to-one

(NEHU, 2016, 2012, 2006, 2003)

Solution: Since $f:X \to Y$ and $g:Y \to Z$ then $gof: X \to Z$

(i) Suppose gof is onto. Then $\forall$ $z \in Z$ $\exists$ $x \in X$ such that $(gof)(x) = z$

We now prove that g is onto.

Since $f:X \to Y$ is a mapping (function) then for each $x \in X$, $\exists$ $y \in Y$ such that $y = f(x)$

Now $z = (gof)(x) = g(f(x)) = g(y)$

Hence $\forall\, z \in Z\ \exists\ y \in Y$ such that $g(y) = z$

$\therefore$ g is onto.

(ii) Suppose gof is one-to-one. Then $\forall\, x_1, x_2 \in X$

$(gof)\,(x_1) = (gof)\,(x_2) \Rightarrow x_1 = x_2$

Let $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$

$\Rightarrow g(f(x_1)) = g(f(x_2))$

$\Rightarrow (gof)\,(x_1) = (gof)\,(x_2)$

$\Rightarrow x_1 = x_2$ [$\because$ gof is one-one]

Hence $\forall\, x_1, x_2 \in X$ such that $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$\therefore$ f is one one

Example 18: Let $f{:}X \to Y$ and $g{:}Y \to X$. Then prove that g is the inverse function of f i.e $g = f^{-1}$, if the composite functions $(gof){:}X \to X$ is the identity function on x and $(fog){:}Y \to Y$ is the identity function on Y.

(NEHU, 2015, 2008, 2007, 2004, 2001)

Solution: The proof follows from theorem 1, theorem 2 and theorem 3.

Example 14: If A is a finite set, then show that a mapping $f{:}A \to A$ is one-one if and only if f is onto. (NEHU, 2012, 2006, 2005, 2014, 2015)

Solution: Let $A = \{a_1, a_2, a_3, \ldots a_n\}$ be finite set of distinct elements.

Suppose f is one-one then

$f(a_1), f(a_2), \ldots\ldots f(a_n)$ are distinct elements of A

$\therefore \{f(a_1), f(a_2), \ldots\ldots f(a_n)\} = \{a_1, a_2, \ldots\ldots a_n\} = A$. (set of distinct elements)

$\Rightarrow$ Range (f) = A

Hence f is onto.

Conversely suppose $f{:}A \to A$ is onto then

Range (f) = A

$\Rightarrow \{f(a_1), f(a_2), \ldots\ldots f(a_n)\} = A = \{a_1, a_2, \ldots\ldots a_n\}$

$\Rightarrow f(a_1), f(a_2), f(a_3), \ldots\ldots f(a_n)$ are distinct elements of A

$\Rightarrow f{:}A \to A$ is one-one

Examplee 19: If A and B be two set and $f{:}A \to B$ is one-one and onto, prove that $f^{-1}{:}B \to A$ is also one-one and onto. (NEHU, 2007, 2002)

Solution: Since $f{:}A \to B$ is one one, hence $\forall\, x_1, x_2 \in A$

$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

Also $f:A \rightarrow B$ is onto, hence $\forall\ y \in B\ \exists\ x \in A$ such that $y = f(x)$

Since f is a bijective function, therefore $f^{-1}:B \rightarrow A$ exists. We now prove that f^{-1} is one-one and onto

(i) f^{-1} is one-one

Let $y_1, y_2 \in B$ such that $f^{-1}(y_1) = f^{-1}(y_2)$

$$\Rightarrow x_1 = x_2 \ \because y_1 = f(x_1),\ y_2 = f(x_2)$$

$$\Rightarrow f(x_1) = f(x_2) \quad \text{since f is one-one}$$

$$\Rightarrow y_1 = y_2$$

Hence f^{-1} is one-one

(ii) f^{-1} is onto.

Since $f:A \rightarrow B$ is onto. Then $\forall\ y \in B\ \exists\ x \in A$ such that $y = f(x)$

$$\Rightarrow f^{-1}(y) = f^{-1}(f(x))$$

$$\Rightarrow f^{-1}(y) = (f^{-1} f)(x)$$

$$\Rightarrow f^{-1}(y) = I_A(x)$$

$$\Rightarrow f^{-1}(y) = x$$

Hence $\forall\ x \in A,\ \exists\ y \in B$ such that $f^{-1}(y) = x$

$\therefore f^{-1}$ is onto

2.14 Graph of a Function

Definition: If $y = f(x)$ be the given function, then the set of points $(x, f(x))$ is said tobe the graph (or curve) of the function.

We plot some of the points and by joining them, we draw the required graph.

Example 1: The graph of $y = \dfrac{x^2}{x}$

Thus can be broken up into $y=x$ and $x \neq 0$; y is undefined for $x = 0$. The values are:

x =	1	2	3	−1	−2	−3	0		$\frac{1}{100}$	$-\frac{1}{100}$
y =	1	2	3	−1	−2	−3	undefined		$\frac{1}{100}$	$-\frac{1}{100}$

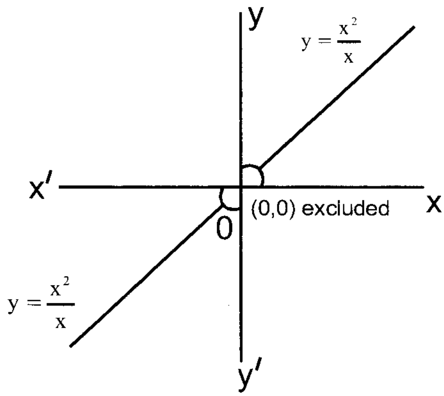


Fig. 1

Example 2: Draw the graph of $y = |x|$

$$y = |x| \text{ means } y = x \text{ for } x>0$$
$$= -x \text{ for } x<0$$
$$= 0 \text{ when } x=0$$

The values are:

x =	0	1	2	–1	–2	–3	$\frac{1}{100}$	$-\frac{1}{100}$
y =	1	2	3	1	2	3	$\frac{1}{100}$	$\frac{1}{100}$

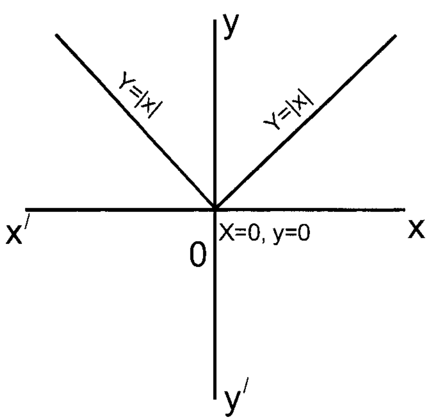


Fig. 2

Example 3: The graph of y =[x] where [x] denotes the greatest integer less than or equal to x.

$$y = [x] \text{ means } y = 0 \text{ for } 0 \leq x < 1$$
$$= 1 \text{ for } 1 \leq x < 2$$
$$= 2 \text{ for } 2 \leq x < 3$$
$$= -1 \text{ for } -1 \leq x < 0$$
$$= -2 \text{ for } -2 \leq x < 1 \text{ and so on}$$

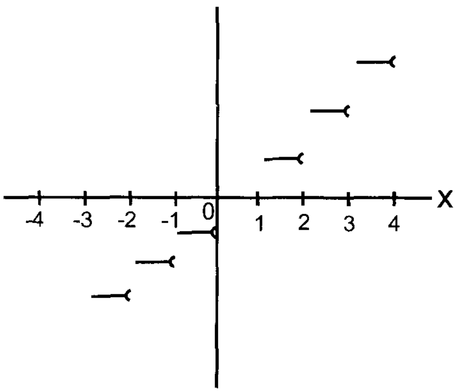


Fig. 3

Example 4: The graph of $y = x^2$

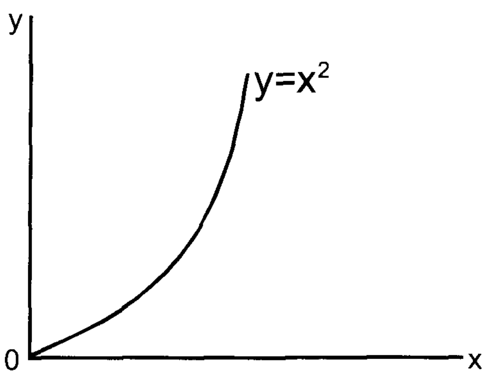


Fig. 4

Example 5: The graph of $y = \sqrt{x}$

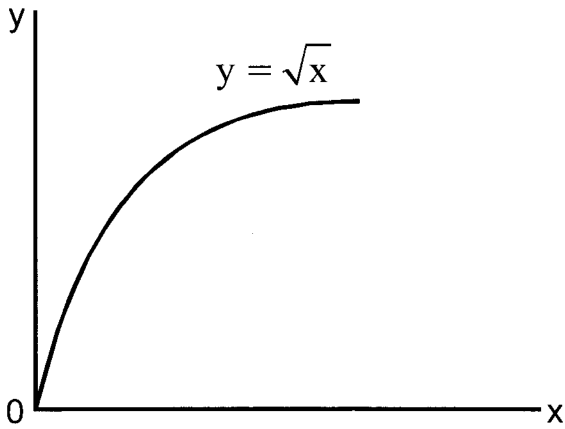


Fig. 5

Example 6: The grpah of $y = e^x$

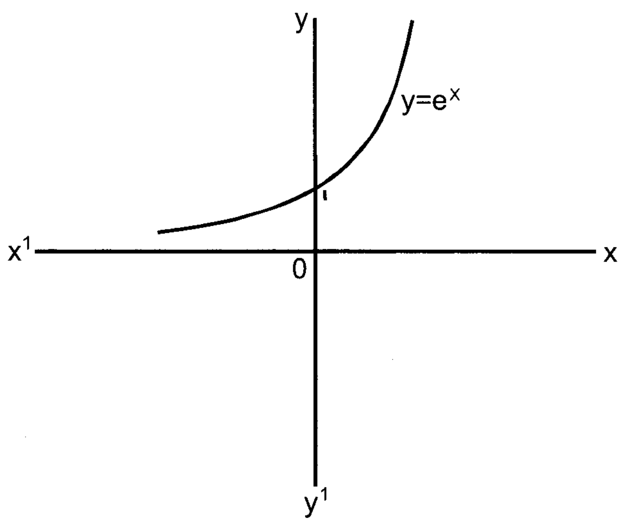


Fig. 6

Example 7: The graph of y = $\log_e^x$

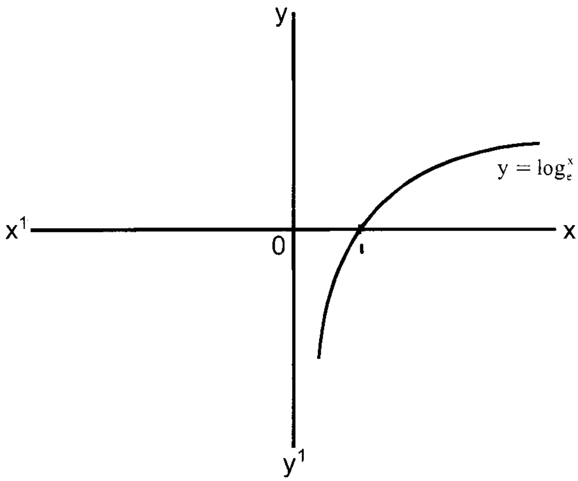


Fig. 7

Example 8: The graph of hyperbolic functions

(i) Sinhx (ii) coshx (iii) tanhx

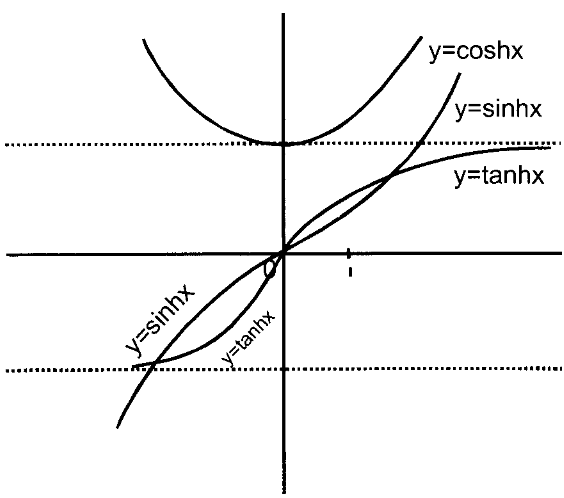


Fig. 8

2.15. Other Types of Functions

(A) Polynomials: A function of the form $f(x) = a_0 + a_1x + a_2x^2 + ... + a_{n-1}x^{n-1} + a_nx^n$ where $a_0, a_1, ...a_n$ are constants and n is a positive integer is said to be a polynomial in x (of degree n)

Examples: $f(x) = x^2$, $f(x) = 2x^3 + 3x^2 + x + 1$ are examples of polynomial functions

(B) Rational Functions: A function which is the quotient of two polynomial functions i.e functions of the form

$$f(x) = \frac{p(x)}{q(x)} = \frac{a_0 + a_1x + a_2x^2 + ... + a_nx^n}{b_0 + b_1x + b_2x^2 + ... + b_mx^m}$$

where $q(x) \neq 0$ is called a rational function.

In particular, when q(x) = constant, f(x) reduces to a polynomial. A ratioinal function is even if both p(x) and q(x) are even or odd functions. It is odd if one is even and the other odd.

(C) Transcendental Functions: Functions like e^x, $\log_e^x$, cosx, sinx, $\sin^{-1}x$, $\cos^{-1}x$, sinhx, coshx, etc. are called Transcendental functions.

(D) Monotonic Functions: Let the function f(x) be defined in the domain D and $x_1, x_2 \in D$. Then f(x) is said to be monotonic

(i) increasing in D if $x_2 > x_1 \Rightarrow f(x_2) \geq f(x_1)$

(ii) decreasing in D if $x_2 > x_1 \Rightarrow f(x_2) \leq f(x_1)$

(iii) strictly increasing in D if $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$

(iv) strictly decreasing in D if $x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$

A function which conforms to any of the above cases is called a monotonic or monotone function.

Example 1: $f(x) = x + 2$ is a monotonic increasing function in IR; since for $x_2 > x_1$, $f(x_2) = x_2 + 2 > x_1 + 2 = f(x_1)$

Example 2: $f(x) = \frac{1}{x+1}$ is a monotonic decreasing function in [0, 100] since as x increases from 0 to 100 f(x) decreases.

(E) Bounded Function: A function f(x) defined on a set D (or domain D) is said to be bounded above on D, if there exists a number M such that $f(x) \leq M \ \forall \ x \in D$

Similarly, a function f(x) defined on a set D (or domain D) is said to be bounded below on D if there exists a number m such that $m \leq f(x) \forall \ x \in D$

The function f(x) is said to be bounded on D if it is bounded both and below i.e. if there exists two real numbers m and M such that

$m \leq f(x) \leq M \ \forall \ x \in D$

m is called the lower bound of f(x) in D and M is called the upper bound of f(x) in D.

Example: f(x) x^2 is bounded in [1,2] since for $1 \leq x \leq 2$, $1^2 = 1 \leq f(x) \leq 2^2 = 4$

Exercises 2.2

1. Defined a function what do you mean by the domain and range of a function? Give examples.
2. Define and give example of each type of the following (i) injective function (ii) surjective function (iii) bijective functions (iv) many-one function (v) into function.
3. Show that the function $f:\mathbb{R} \to \mathbb{R} : f(x) = 1+x^2$ is many-one into.
4. Show that the function $f:\mathbb{R} \to \mathbb{R} : f(x) = x^4$ is many-one and into.
5. Show that the function $f:\mathbb{R} \to \mathbb{R} : f(x) = x^5$ is one-one and onto.
6. Show that the function

   (i) $f:\mathbb{N} \to \mathbb{N} : f(x) = x^2$ is one-one and into

   (i) $f:\mathbb{Z} \to \mathbb{Z} : f(x) = x^2$ is many and into
7. Let $f:[0, \pi/2] \to \mathbb{R}: f(x) = \sin x$ and $g:[0, \pi/2] \to \mathbb{R}: g(x) = \cos x$. Show that each one of f and g is one-one but (f+g) is not one-one.
8. Show that function $f:\mathbb{N} \to \mathbb{Z}$, defined by

$$f(x) = \begin{cases} \frac{1}{2}(x-1), \text{ when x is odd} \\ -\frac{1}{2}x, \text{ when x is even} \end{cases}$$

   is both one-one and onto
9. Show that the function $f:\mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, \text{ if x is rational} \\ -1, \text{ if x is irrational} \end{cases}$$

   is many-one, into. Find (i) $f\left(\frac{1}{2}\right)$ (ii) $f\left(\sqrt{2}\right)$ (iii) $f(\pi)$

10. Find the domain and range of following functions

(i) $f(x) = \frac{x^2}{1+x^2}$ (ii) $f(x) = \frac{x^2+3x+6}{x^2-5x+6}$

(iii) $f(x) = \sqrt{x-1}$ (iv) $f(x) = \log(x^2 - 5x + 6)$

(v) $f(x) = \sqrt{x^2-3x+2}$ (vi) $f(x) = \sqrt{8x+2x-3x^2}$

(v) $f(x) = \frac{1}{\sqrt{(x-2)(4-x)}}$ (viii) $f(x) = \frac{|x|}{x}$

11. Which of the given functions is (are) even, odd and which is (are) neither even nor odd?

(i) $f(x) = \sin x$ (ii) $f(x) = \sin x + \cos x$

(iii) $f(x) = 7$ (iv) $f(x) = x^2 - |x|$

12. Prove that any function of x, defined for all real x, is the sum of an even and an odd function of x.

[Hints: $f(x) = \frac{1}{2}[\{f(x) + f(-x) + \{f(x) - f(-x)\}$

13. If $f(x) = \frac{1+x}{1-x}$ then prove that $2f(x).f(x^2) = 1+\{f(x)\}^2$ (NEHU, 2014)

14. Let $A = \{x \in \mathbb{R} : 0 \le x \le 1\}$. If $f:A \to A$ is defined by

$$f(x) = \begin{cases} x, \text{if } x \in Q \\ 1-x, \text{if } x \notin Q \end{cases}$$

Then prove that $(f:f)(x) = x \ \forall \ x \in A$ (NEHU, 2013)

15. What do you mean by a bijective map? Show whether the mapping $f:\mathbb{N} \to \mathbb{N}$ defined by $f(n) = n-(-1)^n$, $n \in \mathbb{N}$ is bijective, where $\mathbb{N}$ is the set of natural numbers. (NEHU, 2011)

16. Sketch the graph of the following functions

(i) $f(x) = 2$, whenx is an integer

$= 0$, when x is not an integer

(ii) $f(x) = 0$, for $x^2 > 1$

$= 1$, for $x^2 < 1$

$= \frac{1}{2}$, for $x^2 = 1$

(iii) $f(x) = 0$, when x is an integer

$= 2x$, when x is not an integer

(iv) $f(x) = x - [x]$ where [x] denotes the greatest integer not greater than x

17. A taxi company charges one rupee for one kilometre or less from start and at the rate of 50 paise per kilometre or any fraction thereof for additional distance. Express analytically the fare 'f' in rupee as a function of the distance 'd' in kilometre and draw the graph of the function.

(NEHU, 2001)

18. Let A = { 2, 3, 4, 5} and B = {7, 9, 11, 13} and let f = {(2,7), (3, 9), (4, 11), (5, 13)}. Show that f is invertible and find f^{-1}

19. Let $f:Q\to Q : f(x) = 3x-4$. Show that f is invertible and find f^{-1}.

20. Show that the function $f:IR\to IR : f(x) = 2x+3$ is invertible and find f^{-1}.

21. Let $f:IR\to IR : f(x) = \frac{1}{2}(3x+1)$. Show that f is invertible and find f^{-1}.

22. If $f(x) = \frac{4x+3}{6x-4}, x\neq\frac{2}{3}$. Show that (fof) x = x for all $x\neq\frac{2}{3}$. Hence find f^{-1}.

23. Let IR_+ be the set of all non negative real numbers. Show that the function $f:IR_+\to[-5, \infty[$ defined by $f(x) = 9x^2 + 6x - 5$ is invertible and find f^{-1}.

24. Show that $f(x) = 2x^2 + 4x + 6$ in the internal [0, 1] is bounded and has lower bound 6 and upper bound 12.

25. Show that $f(x) = \frac{x}{x+1}$ is monotonic ascending for $x>0$.

26. Show that the function $f(x) = \frac{3x+5}{2x+1}$ is a strictly decreasing function.

27. (i) If $f(x) = (x-a)(x-b)(x-c)$, show that values of f(a), f(b), f(c) and f(0) are respectively 0, 0, 0 and $-abc$

(ii) If $f(x) = e^x$, show that $f(a), f(b) = f(a+b)$

(iii) If $f(x) = x + |x|$. Are f(3) and f(–3) equal?

(iv) If $f(x) = \frac{x}{1-x}$, show that $\frac{f(x+h)-f(x)}{h} = \frac{1}{(1-x)(1-x-h)}$

28. When $f(x) = \log\sin x$, $\phi(x) = \log\cos x$, verify that

(i) $f(x) + \phi(x) + \log 2 = f(2x)$

(ii) $e^{2o(x)} = \frac{1}{2}(1+\cos 2x)$

(iii) $c^{2f(x)} + e^{2o(x)} = 1$

29. (i) If $\phi(x) = m\,\frac{x-l}{m-l} + l\,\frac{x-m}{l-m}$, show that $\phi(l) + \phi(m) = \phi(l+m)$

(ii) Verify that $\frac{f(x)-f(y)}{1+f(x).f(y)} = \frac{x-y}{1+xy}$ when $f(x) = \frac{x-1}{x+1}$

(iii) When $y = f(x) = \frac{lx+m}{nx-l}$, prove that $f(y) = x$

(iv) If f(x)= min {x, $1/x$} for x>0; find f(3) and f($1/3$). For a>0, is it true that f(a) = f($1/a$)? Can f(–4) be found out?

(v) If $f(x) = \frac{|x|}{x}$ and $c \neq 0$, be real number, show that $|f(c) - f(-c)| = 2$

3

Limit of a Function

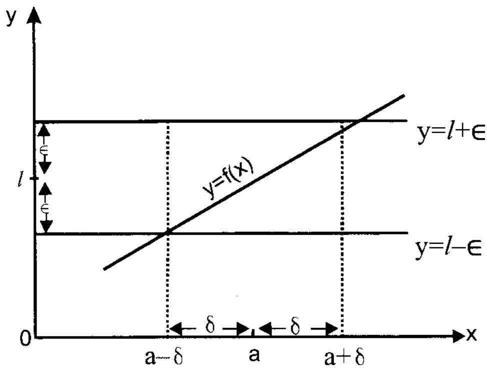
Introduction

The concept of limit forms the most outstanding concept in Calculus and plays an important role in the development of this subject. It is this concept of limit which make the line of difference of calculus with Algebra, the latter being based upon the four fundamental operations of addition, subtraction, multiplication and division. The real essence and strength of this subject lies in the concept of limit *upon which the new and broad structure of calculus.*

3.1 Difiniton

When x approaches a constant quantity 'a' from either side (but$\neq$a), if there exists a definite finite number 'l' towards which f(x) approaches [As a particular case f(x) may remain equal to l when x is sufficiently close to a] such that the numerical difference of f(x) can be made as small as we please (i.e less than any pre-assigned prositve quantity however small) by taking x sufficiently close to 'a', then 'l' is defined as the limit of f(x) as x approaches 'a' and is symbolically written as $\lim_{x \to a} f(x) = l$ or $f(x) \to l$ as $x \to a$.

Mathematically, f(x) is said to have a limit 'l' (where $l \in \mathbb{R}$) as $x \to a$, if for any pre assigned positive quantity $\in$ however it may be, there exists a positive number δ (depending on $\in$) such that $|f(x)-l| < \in$ when $0<|x-a|<\delta$. Geometrcally we may say that for all x in the two open intervals $a-\delta<x<a$ and $a<x<a+\delta$, the graph of f(x) lies between the horizontal lines $y=l-\in$ and $y=l+\in$.



Observations

1. Here the graph of y = f(x) has been assumed to be without any break in the interval considered. In the determination of the limit of y= f(x) as $x \to a$, we are not at all concerned with the point on the graph corresponding to x=a. The point on the graph corresponding to x=a may not belong to the curve or even may not exists at all.

2. If there eixsts a number l such that $\lim_{x \to a} f(x) = l$ we say that $\lim_{x \to a} f(x)$ exists. However if no such l exists, we say that $\lim_{x \to a} f(x)$ does not exists. A limit, if exists, is necessary unique.

3. (a) The definition requires that for every $\in > 0$, however small, there exists a suitable δ. Thus, for each value of $\in$, a largest permissible value of δ is determined and so δ is a function of $\in$ [which is sometimes denoted by $\delta(\in)$], it is however clear, that for a fixed 'a', the smaller the given value of $\in$, the smaller the value of δ.

   (b) In order to prove the existence of the limit of f(x), from the above definition, it is sufficient if we can show that the inequality $0 < |x-a| < \delta$ follows from the inequality $|f(x)-l| < \in$, $\in$ being given in advance and thus δ can be obtained.

4. It is important to note that 'a' need not be in the domain of f(x). The limit has nothing to do with the value of f(x) at x=a itself, and, infact, f(a) need not even be defined. The existence of the limit of f(x) depends entirely on the values of f(x) for x near a (not for x at a)

Example 1: Show that $\lim_{x \to 2} 5x = 10$

Solution: Let $\in > 0$ be given (however small), we are to find a value of δ depending on the $\in$ such that

$|5x-10| < \epsilon$ for $0<|x-2|<\delta$

i.e $|x-2| < \epsilon/5$ for $0<|x-2|<\delta$

taking $\delta = \epsilon/5$ we see that $0<|x-|<\delta$ follows from $|5x-10|<\epsilon$.

Therefore the existence of the limit is assured by definition.

Example 2: Show that $\lim_{x\to 3}\dfrac{x^2-9}{x-3} = 6$

Solution: Here we are to find a $\delta>0$, depending on any given $\epsilon>0$ (however small) such that

$$\left|\frac{x^2-9}{x-3}-6\right|<\epsilon \text{ for } 0 < |x-3| < \delta$$

i.e $$\left|\frac{(x-3)(x+3)}{x-3}-6\right|<\epsilon \text{ for } 0 < |x-3| < \delta$$

i.e $|x+3-6| < \epsilon$ for $0 < |x-3| < \delta$ $[\because x\to 3\ x-3\neq 0]$

i.e $|x-3| < \epsilon$ for $0 < |x-3| < \delta$

Taking $\delta=\epsilon$, we see that the relations are satisfied and hence by definition, the limit exists.

3.2 One-sided limits: Right-hand and left hand limits.

A function f(x) is said to have a limit l_1 (say) as $x\to a$ from the left, if for every $\epsilon>0$; we can find a number $\delta>0$ such that $|f(x)-l_1| < \epsilon$ for $a-\delta<x<a$ and it is written as $L\lim_{x\to a} f(x) = l_1$ or $\lim_{x\to a-0} f(x) = l_1$ or $f(a-0) = l_1$ and l_1 as called the left hand limit of f(x).

Similarly, a function f(x) is said to have a limit l_2 (say) as $x\to a$ from the right, if for every $\epsilon>0$, we can find a number $\delta>0$ such that $|f(x) - l_2|<\epsilon$ for $a<x<a+\delta$ and it is written as $R\lim_{x\to a} f(x) = l_2$ or $\lim_{x\to a+0} f(x) = l_2$ or $f(a+0) = l_2$ and l_2 as called the left hand limit of f(x).

Hence at once it follows from the definition that $\lim_{x\to a} f(x) = l$ if only if

$\lim_{x\to a-0} f(x) = l = \lim_{x\to a+0} f(x)$.

Observation

1. One sided limit are very useful is solving different types of problems particularly at the end points of the domain of definition of a function.

2. If either $\lim\limits_{x\to a-0} f(x)$ or $\lim\limits_{x\to a+0} f(x)$ fails to exist, then $\lim\limits_{x\to a} f(x)$ is said to be non existence

Example 1: Let $f(x) = \dfrac{x^2}{x}$, find $\lim\limits_{x\to a} f(x)$ if it exists.

Solution: Intuitive approach: We see that $\dfrac{x^2}{x} = x$, when $x \neq 0$

$$\text{Now } \lim_{x\to 0-0} f(x) = \lim_{x\to 0-0} \frac{x^2}{x} = \lim_{x\to 0-0} x = 0$$

$$\lim_{x\to 0+0} f(x) = \lim_{x\to 0-0} \frac{x^2}{x} = \lim_{x\to 0-0} x = 0$$

Hence $\lim\limits_{x\to 0} f(x)$ exists and $\lim\limits_{x\to 0} f(x) = 0$

Analytic Approach:

We are to find a value of δ depending on any given $\in > 0$ such that

$$\left|\frac{x^2}{x} - 0\right| < \in \text{ for } 0 < |x-0| < \delta$$

i.e for $|x| < \in$ for $0 < |x-0| < \delta$

Taking $\delta = \in$ we see that the relations are satisfied and hence existence of the limit to be 0 is assured.

Example 2: Find $\lim\limits_{x\to a} \dfrac{|x|}{x}$ if it exists

Solution: We know that $|x| = x$ when $x \geq 0 = -x$ when $x < 0$

$$\text{Now } \lim_{x\to 0-0} \frac{|x|}{x} = \lim_{x\to 0-0} \frac{-x}{x} = \lim_{x\to 0-0} (-1) = -1$$

$$\text{and } \lim_{x\to 0-0} \frac{|x|}{x} = \lim_{x\to 0-0} \frac{x}{x} = \lim_{x\to 0-0} (1) = 1$$

Hence $\lim\limits_{x\to 0} \dfrac{|x|}{x}$ does not exists

3.3 Distinction between $\lim\limits_{x\to a} f(x)$ and f(a)

$\lim\limits_{x\to a} f(x)$ is the value of f(x) when x has any value arbitrarily near to a except

a. In this case, we do not care to know what happens when x is put equal to a. But f(a) stands for the value of f(x) when x is exactly equal to a obtained either by the definition of the function at a, or else by substitution of a for x in the expression f(x), when it exists.

3.4 Different Types of Limits

(I) Function tending to infinity: $\lim_{x\to a} f(x) = \pm\infty$

A function f(x) is said to tend to ∞ (or $-\infty$) as $x \to a$ if for any pre assigned positive quantity N, however large, we can find a positive number δ, such that f(x) > N (or – f(x) > N) for $0 < |x-a| < \delta$ and it is written as $\lim_{x\to a} f(x) = \infty$ (or $\lim_{x\to a} f(x) = -\infty$)

Example: Show that $\lim_{x\to 0} \frac{1}{x^2} = \infty$

Solution: Here we must find a $\delta > 0$ as a function of N

such that $\frac{1}{x^2} > N$ for $0 < |x-0| < \delta$

i.e $x^2 < \frac{1}{n}$ for $0 < |x| < \delta$

Taking $\delta = \frac{1}{\sqrt{N}}$ the result follows

(II) Limit of a function as the variable tends to infinity: $\lim_{x\to\infty} f(x)$

A function f(x) is said to have a limit l as $x \to \infty$ or $(-x \to \infty)$ if for any pre assigned positive number ϵ, however small it may be, there corresponds a positive number m, however large such that

$|f(x) - l| < \epsilon$ for x>m (or – x>m) and is written as $\lim_{x\to\infty} f(x) = l$ (or $\lim_{x\to -\infty} f(x) = l$)

Example: Show that $\lim_{x\to\infty} \frac{1}{x^2} = 0$

Solution: We are to find n as a function of $\epsilon > 0$ such that $\left|\frac{1}{x^2} - 0\right| < \epsilon$ for x>m i.e $\frac{1}{x^2} < \epsilon$ for x>m

i.e $x^2 < \frac{1}{\epsilon}$ for x>m

i.e $x > \frac{1}{\sqrt{\epsilon}}$ (or $x < -\frac{1}{\sqrt{\epsilon}}$) for x>m

Taking $m = \frac{1}{\sqrt{\epsilon}}$ both the relations are satisfied.

Hence the result.

(III) Function which tends to infinity as the variable tends to infinity:

$\lim_{x\to\infty} f(x) = \infty$ or $\lim_{x\to-\infty} f(x) = \infty$

A function f(x) is said to tend to ∞ as $x\to\infty$ (or $-x\to\infty$) if for any pre assigned positive number N, there exists a positive number m such that f(x)>N for x > m (or – x>m) and is written as

$\lim_{x\to\infty} f(x) = \infty$ or $\lim_{x\to-\infty} f(x) = \infty$

Example: Show that $\lim_{x\to\infty} x^2 = \infty$

Solution: We are to find m corresponding to N such that $x^2>N$ for x>m

i.e $x>\sqrt{N}$ (or $x<-\sqrt{N}$) for x>m

Taking $m = \sqrt{N}$ we see that $\lim_{x\to\infty} x^2 = \infty$

3.5 Cauchy Criterion for existence of a limit of a function.

$\lim_{x\to a} f(x)$ exists, if corresponding to any given positive number ϵ, however small it may be, there exists a positive number δ, such that $|f(x_1) - f(x_2)|<\epsilon$ for every pair x_1, x_2 of values of x satisfying $0<|x_1-a| < \delta$ and $0<|x_2-a|<\delta$

In otherwords, this Cauchy condition is sometimes writte as

$$\lim_{\substack{x_1\to a\\ x_2\to a}} |f(x_1) - f(x_2)| = 0$$

Theorem: Cauchy's general principle of existence of limit.

The necessary and sufficient condition that a function f(x) may tend to a definite finite limit as $x \to a$, is that to any pre assigned positive quantity $\in$, however, small, there corresponds a positive number δ such that $|f(x_2)-f(x_1)| < \in$ for every pair x_1, x_2 of values of x satisfying $0<|x_1-a|<\delta$ and $0<|x_2-a|<\delta$.

The proof of this theorem is beyond the scope of this book.

Example: Does $\lim\limits_{x\to 0} \sin \frac{1}{x}$ exists?

Solution: Suppose $\lim\limits_{x\to 0} \sin \frac{1}{x}$ exists.

Then by Cauchy's general principle of existence of limit (above theorem), to any pre assigned positive quantity $\in$, however small there corresponds a positive number δ such that for every pair x_1, x_2 of values of x

$$\left|\text{Sin}\frac{1}{x^2} - \text{Sin}\frac{1}{x^1}\right| < \in \text{ for } 0 < |x_1| < \delta \text{ and } 0 < |x_2| < \delta$$

But if we take $x_2 = \dfrac{2}{(4n+1)\pi}$ and $x_1 = \dfrac{2}{(4n-1)\pi}$ where $x \in \mathbb{Z}$ we see that $0 < |x_1| < \delta$ and $0 < |x_2| < \delta$ by choosing n sufficiently large.

However $\left|\text{Sin}\frac{1}{x^2} - \text{Sin}\frac{1}{x^1}\right| = \left|\text{Sin}\left(2n+\frac{1}{2}\right)\pi - \text{Sin}\left(2n-\frac{1}{2}\right)\pi\right| = 2 \nless$ any $\in$

Hence our assumption is wrong and $\lim\limits_{x\to 0} \sin \frac{1}{x}$ does not exists.

3.6 Fundamental Theorems on Limits

Theorem 1: If $\lim\limits_{x\to a} f(x) = l$ and $\lim\limits_{x\to a} g(x) = m$, then $\lim\limits_{x\to a} \{f(x) \pm g(x)\} = l \pm m$

i.e. The limit of the sum or difference of two functions is equal to the sum or difference of their limits

Proof: Let $\in>0$ be given.

Since $\lim\limits_{x\to a} f(x) = l$ and $\lim\limits_{x\to a} g(x) = m$.

Then to this given $\in>0$, there corresponds two positive numbers δ_1 and δ_2 such that

$$|f(x) - l| < \in/2 \text{ for } 0 < |x-a| < \delta_1$$

and $|g(x) - m| < \epsilon/2$ for $0 < |x-a| < \delta_2$

Let $\delta = \min\{\delta_1, \delta_2\}$ then for $0 < |x-a| < \delta$; $|f(x) - l| < \epsilon/2$ and $|g(x) - m| < \epsilon/2$

We have

$|f(x) \pm g(x) - (l \pm m)| \leq |f(x) - l| + |g(x) - m| < \epsilon/2 + \epsilon/2 = \epsilon$

Hence the theorem.

Theorem 2: If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m$, then $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} \{f(x).g(x)\} = l.m$

i.e the limit of the product of two functions is equal to the product of their limits.

Proof: Let $\epsilon > 0$ be given. Then there corresponds poisitive numbers δ_1 and δ_2 such that

$$|f(x) - l| < \epsilon' \text{ for } 0 < |x-a| < \delta_1$$

and $|g(x) - m| < \epsilon''$ for $0 < |x-a| < \delta_2$ ....(1)

Choosing $\delta = \min\{\delta_1 \text{ and } \delta_2\}$ then for $0 < |x-a| < \delta$ we have

$|f(x).g(x) - l.m| = |g(x)\{f(x) - l\} + l\{g(x) - m\}|$

$\Rightarrow |f(x)\, g(x) - lm\} \leq |g(x)|\, |f(x) - l| + |l|\, |g(x) - m|$

$$< \{|m| + \epsilon\}.\epsilon' + |l|.\epsilon' \text{ from (1)}$$

$$= \{|m| + |l| + \epsilon''\}\, \epsilon'$$

$$< \epsilon \text{ for } 0 < |x-a| < \delta$$

by choosing ϵ' less than 1 and $< \dfrac{\epsilon}{|m| + |l| + 1}$

Hence the theorem

Theorem 3: If $\lim_{x \to a} f(x) = l$ and $\lim_{x \to a} g(x) = m \neq 0$, then

$$\lim_{x \to a} \left\{\frac{f(x)}{g(x)}\right\} = \frac{l}{m}$$

i.e The limit of the quotient of two functions is equal to the quotient of their limits, provided that the limit of the denominator is not zero.

Proof: Consider $\left|\frac{f(x)}{g(x)}-\frac{l}{m}\right| = \left|\frac{mf(x)-lg(x)}{mg(x)}\right|$

$$= \left|\frac{m\{f(x)-l\}-l\{g(x)-m\}}{mg(x)}\right|$$

$$\leq \frac{|m||\{f(x)-l|+|l||g(x)-m|}{|m||g(x)|} \quad ...(i)$$

Since $\lim\limits_{x\to a} f(x) = l$ and $\lim\limits_{x\to a} g(x) = m \neq 0$, to a given $\in>0$, there corresponds positive numbers δ_1 and δ_2 such that

$$|f(x) - l| < \in' \text{ for } 0<|x-a| < \delta$$

and $|g(x) - m\} < \frac{1}{2}|m| < \in'$ for $0<|x-0| < \delta_2$

i.e $|m| - |g(x)| \leq |g(x) - m| < \frac{1}{2}|m|$

i.e $|g(x)| > \frac{1}{2}|m|$ for $0<|x-a| < \delta_2$

Hence by taking δ=min (δ_1 and δ_2), for $0<|x-a| < \delta$

We have from (i)

$$\left|\frac{f(x)}{g(x)}-\frac{l}{m}\right| < \frac{|m|\in'+|l|\in'}{|m|.\frac{1}{2}|m|} = \frac{2(|m|+|l|)\in'}{|m|^2} < \in$$

by choosing $\in' < \frac{|m|^2 \in}{2(|m|+|l|)}$, the theorem follows.

Example: Evaluate $\lim\limits_{x\to 3} \frac{x^2+x-12}{x-3}$

Solution: Here if we put x=3, the function takes the form % which is not defined.

Also since the limit of the denominator is 0, we cannot apply Theorem 3 directly.

Since x = 3 may be excluded by cancelling the common factor we get

$$\lim_{x\to 3}\frac{x^2+x-12}{x-3} = \lim_{x\to 3}\frac{(x+4)(x-3)}{x-3} = \lim_{x\to 3}(x+4) = 7$$

Theorem 4: Limit of a function of a function.

If $\lim_{x\to a}\phi(x) = b$ and $\lim_{u\to a} f(u) = f(b)$, then $\lim_{x\to a} f\{\phi(x)\} = f\{\lim_{x\to a}\phi(x)\}$

Proof: Since $\lim_{u\to a} f(u) = f(b)$, then for any given $\in>0$, there corresponds a positive number δ_1 $\lim_{x\to a} f\{\phi(x)\}$ such that $|f\{\phi(x)\} - f(b)| < \in$ when $0<|\phi(x)-b|<\delta_1$. Again since $\lim_{x\to a}\phi(x) = b$, therefore, corresponds to this positive number δ_1, there exists a positive number δ such that $|\phi(x)-b|<\delta_1$ for $0 < |x-a| < \delta$.

Combining these two results, we see that to the given arbitrary positive number $\in$, there corresponds a positive number δ suich that $|f\{\phi(x)\} - f(b)| < \in$ for $0<|x-a| < \delta$

i.e $\lim_{x\to a} f\{\phi(x)\} = f(b) = f\{\lim_{x\to a}\phi(x)\}$

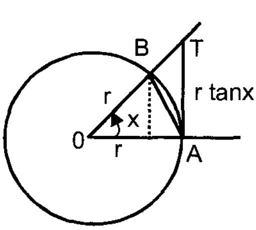
Corrollary:

(i) $\lim_{x\to a}\{\phi(x)\}^{f(x)} = \left\{\lim_{x\to a}\phi(x)\right\}^{\lim_{x\to a} f(x)}$ provided $\phi(x)$ is positive for all values of x

(ii) $\lim_{x\to a}\{f(x)\}^n = \left\{\lim_{x\to a} f(x)\right\}^n$

3.7 Evaluation of Some Important Limits:

1. $\lim_{x\to a}\frac{\sin x}{x} = 1$



Proof: Let $0 < x < \pi/2$

Then from the figure, it is clear that

Area of triangle AOB < Area of the sector AOB < Area of triangle AOT

i.e $\frac{1}{2}.r.r\sin x < \frac{1}{2} r^2.x < \frac{1}{2}$ r. rtanx i.e $\sin x < x < \tan x$

i.e $1 < \dfrac{x}{\sin x} < \dfrac{1}{\cos x}$

Let $x \to 0+0$, then $\cos x \to 1$ and $\dfrac{1}{\cos x} \to 1$

Hence $\lim\limits_{x\to 0+0} \dfrac{x}{\sin x} = 1$ and therefore $\lim\limits_{x\to 0+0} \dfrac{\sin x}{x} = 1$

Since $\dfrac{\sin(-x)}{-x} = \dfrac{\sin x}{x}$

Hence no further discussion is required to proof that $\lim\limits_{x\to 0-0} \dfrac{\sin x}{x}$ and therefore $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$

2. $\lim\limits_{x\to\infty} \left(1+\dfrac{1}{x}\right)^x = e$

Proof: Let x be any large positive real number. Then without any loss, there are two consecutive positive integers n and n+1 such that

$n \le x < n+1$

$\therefore\ 1+\dfrac{1}{n} \ge 1+\dfrac{1}{x} > 1+\dfrac{1}{n+1}$

Since each being greater than 1 and as $n+1 > x \ge n$

$$\left(1+\frac{1}{n}\right)^{n-1} > \left(1+\frac{1}{x}\right)^{x} > \left(1+\frac{1}{n+1}\right)^{n}$$

or $\left(1+\dfrac{1}{n}\right)\left(1+\dfrac{1}{n}\right)^{n} > \left(1+\dfrac{1}{x}\right)^{x} > \left(1+\dfrac{1}{n+1}\right)^{n-1} \Big/ \left(1+\dfrac{1}{n+1}\right)$

Now when $x \to \infty$, $n \to \infty$ and 'n' being a positive integer we have both $\left(1+\dfrac{1}{n}\right)^{n}$ and $\left(1+\dfrac{1}{n}\right)^{n-1} \to e$

Also $1+\frac{1}{n} \to 1$ and $1+\frac{1}{n+1} \to 1$

Hence the two extremes in the above inequality tend to a common limit e

and hence $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

Let x be any negative real number

i.e $x = -p$ where p is a large positive real number

Then as $p\to\infty$, $x\to-\infty$

$$\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = \lim_{p\to\infty}\left(1-\frac{1}{p}\right)^{-p} = \lim_{p\to\infty}\left(\frac{p}{p-1}\right)^p$$

$$= \lim_{p\to\infty}\left(1+\frac{1}{p-1}\right)^p$$

$$= \lim_{p\to\infty}\left(1+\frac{1}{p-1}\right)^{p-1} . \lim_{p\to\infty}\left(1+\frac{1}{p-1}\right)$$

$$= e.1=e$$

Hence for all real values of x $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$

Corollary: $\lim_{x\to 0}(1+x)^{1/x} = e$

Proof: Putting $x = \frac{1}{y}$ in 1 and the result follows

3. $\lim_{x\to 0}\frac{1}{x}\log(1+x)=1$

Proof: We have

$$\lim_{x\to 0}\frac{1}{x}\log(1+x) = \lim_{x\to 0}\log(1+x)^{\frac{1}{x}}$$

$$= \log\left\{\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right\} \text{ by theorem 4}$$

$$= \log e = 1$$

4. $\lim_{x\to 0}\frac{a^x-1}{x}=\log_e^a\ (a>0)$

Proof: Putting $a^x-1 = y$ so that $y\to 0$ as $x\to 0$ and $a^x=1+y$ or $x \log a = \log(1+y)$

$$\text{Hence } \lim_{x\to 0}\frac{a^x-1}{x} = \lim_{y\to 0}\frac{y}{\log(1+y)}.\log a$$

$$= \log a.\lim_{y\to 0}\frac{y}{\frac{1}{y}\log(1+y)}$$

$$= \log a.\frac{1}{\log e} \text{ by (3)}$$

$$= \log a$$

5. $\lim_{x\to 0}\frac{e^x-1}{x}=1$

Proof: Take $e^x-1 = y$ and proceed as above

6. $\lim_{x\to a}\frac{x^n-a^n}{x-a} = n\,a^{n-1}$ for $a>0$ and n a rational number.

Proof:

Case 1: When n is a positive integer

By actual division we have

$$\frac{x^n-a^n}{x-a} = x^{n-1} + x^{n-2} + x^{n-3}a^2 + \dots + a^{n-1}$$

$$\therefore \lim_{x\to a}\frac{x^n-a^n}{x-a} = a^{n-1} + a^{n-2}.a + a^{n-3}.a^2 + \dots + a^{n-1}$$

$$= n\,a^{n-1}$$

Case 2: When n is a negative integer

Suppose $n = -m$ where m is a positive integer

$$\text{Then } \frac{x^n-a^n}{x-a} = \frac{x^{-m}-a^{-m}}{x-a} = -\frac{1}{x^m a^m}.\frac{x^m-a^m}{x-a}$$

$$\therefore \lim_{x\to a} \frac{x^n - a^n}{x - a} = \lim_{x\to a} - \frac{1}{x^m a^m} \cdot \frac{x^m - a^m}{x - a}$$

$$= - \frac{1}{a^m . a^m} . \text{ m } a^{m-1} \text{ by case 1.}$$

$$= - m\ a^{-m-1} = n\ a^{n-1}$$

Case 3: When n is a rational fraction

Suppose $n = p/q$, where q is a positive integer and p is any in teger, positive or negative.

Let us put $x^{\frac{1}{q}} = y$ and $a^{\frac{1}{q}} = b$

$$\therefore \lim_{x\to a} \frac{x^n - a^n}{x - a} = \frac{x^{p/q} - a^{p/q}}{x - a} = \frac{y^p - b^p}{y^q - b^q} = \frac{\left(y^p - b^p\right)/\ (y - b)}{\left(y^p - b^p\right)/\ y - b}$$

Now as $x \to a$, $x^{\frac{1}{q}} \to a^{\frac{1}{q}}$ i.e $y \to b$

$$\text{Hence } \lim_{x\to a} \frac{x^n - a^n}{x - a} = \lim_{y\to b} \frac{\left(y^p - b^p\right)/y - b}{\left(y^q - b^q\right)/y - b}$$

$$= \frac{p b^{p-1}}{q b^{q-1}} = \frac{p}{q}\ b^{p-q}$$

$$= \frac{p}{q} \left(a^{1/q}\right)^{p-q} = \frac{p}{q} a^{p/q - 1} = n\ a^{n-1}$$

7. $\lim_{x\to a} \frac{(1+x)^n - 1}{x} = n$

Proof: Put $(1 + x)^n - 1 = y$ so that $y \to 0$ as $x \to 0$ and $(1 + x)^n = 1 + y$
or $n \log (1 + x) = \log (1 + y)$

$$\therefore \lim_{x\to 0} \frac{(1+x)^n - 1}{x} = \lim_{x\to 0} \frac{y}{x}$$

$$= \lim_{x\to 0} \frac{y}{\log(1+y)} \cdot \frac{\log(1+y)}{x}$$

$$= \lim_{x\to 0} \frac{y}{\log(1+y)} \cdot \frac{n\log(1+x)}{x}$$

$$= \lim_{y\to 0} \frac{y}{\log(1+y)} \cdot \lim_{x\to 0} \frac{n\log(1+x)}{x}$$

$$= \lim_{y\to 0} \frac{1}{\frac{1}{y}\log(1+y)} \cdot n \lim_{x\to 0} \frac{1}{x}\log(1+x)$$

$$= \lim_{y\to 0} \frac{1}{\log(1+y)^{1/y}} \cdot n \lim_{x\to 0} \log\ (1+x)^{1/x}$$

$$= 1.n.1 = n \text{ by } (3)$$

Illustrative Examples

Example 1: Apply $(\in, \delta)$ definition to illustrate that $\lim_{x\to 4}(2x-2) = 6$

(NEHU, 2005)

Solution: Let $\in>0$ be given (however small), we are to find a value $\delta>0$ depending an this $\in$ such that

$|(2x-2) - 6| < \in$ for $0 < |x-4| < \delta$

i.e $|2x-8) < \in$ for $0 < |x-4| < \delta$

i.e $|2|\ |x-4) < \in$ for $0 < |x-4| < \delta$

i.e $|x-4| < \in/2$ for $0 < |x-4| < \delta$

Choosing $\delta = \in/2$ we see that $0 < |x-4| < \delta$ follows from $|(2x - 2) - 6| < \in$ and hence the result.

Example 2: Using $(\in, \delta)$ definition, show that $\lim_{x\to 2} \frac{2x^2-8}{x-2} = 8$ (NEHU, 2016)

Solution: Let $\in>0$ be given (however small), we are to find a value $\delta>0$ depending an this $\in$ such that

$$\left|\frac{2x^2-8}{x-2} - 8\right| < \in \text{ for } 0 < |x-2| < \delta$$

i.e $\left|\frac{2x^2-8-8x+16}{x-2}\right| < \in$ for $0 < |x-2| < \delta$

i.e $\left|\frac{2x^2-8x+8}{x-2}\right| < \in$ for $0 < |x-2| < \delta$

i.e $\left|\frac{2(x-2)^2}{x-2}\right| < \in$ for $0 < |x-2| < \delta$

i.e $2|x-2| < \in$ for $0 < |x-2| < \delta$ $[\because x \to 2 \; x \neq 2 \; x-2 \neq 0]$

Choosing $\delta = \in/2$ we see that the relations are satisfied and hence the limit exists.

Example 3: Evaluate the limit $\lim_{x\to 0} \frac{1}{x}\left\{\sqrt{(1+x)}-\sqrt{(1-x)}\right\}$ (NEHU, 2005)

Solution: We remove the indeterminate form 0/0 by transforming

$$\frac{1}{x}\left\{\sqrt{(1+x)}-\sqrt{(1-x)}\right\} = \frac{1}{x}\frac{\left\{\sqrt{(1+x)}-\sqrt{(1-x)}\right\}\times\left\{\sqrt{(1+x)}+\sqrt{(1-x)}\right\}}{\left\{\sqrt{(1+x)}+\sqrt{(1-x)}\right\}}$$

$$= \frac{1}{x}\frac{\left\{\sqrt{(1+x)}\right\}^2-\left\{\sqrt{(1-x)}\right\}^2}{\left\{\sqrt{(1+x)}+\sqrt{(1-x)}\right\}}$$

$$= \frac{1}{x}\frac{2x}{\sqrt{(1+x)}+\sqrt{(1-x)}}$$

$$= \frac{2}{\sqrt{(1+x)}+\sqrt{(1-x)}} \text{ whereby}$$

$$\lim_{x\to 0}\frac{1}{x}\sqrt{(1+x)}-\sqrt{(1-x)} = \lim_{x\to 0}\frac{2}{\sqrt{(1+x)}+\sqrt{(1-x)}} = \frac{2}{2} = 1$$

Example 4: Prove that $\lim_{x\to 0}\frac{e^x-1}{x}=1$ and use the result to evaluate $\lim_{x\to 0}\frac{e^{5x}-1}{7x}$

(NEHU, 2016)

Solution: $\lim\limits_{x\to 1}\dfrac{e^x-1}{x}=1$ has been proved in 3.7 (5)

Now $\lim\limits_{x\to 0}\dfrac{e^{5x}-1}{7x} = \lim\limits_{x\to 0}\dfrac{e^{5x}-1}{\frac{7}{5}5x} = \dfrac{7}{5}\lim\limits_{x\to 0}\dfrac{e^{5x}-1}{5x}$

Put x = 5x, then as $x\to 0$, $z\to 0$

$\therefore \lim\limits_{x\to 0}\dfrac{e^{5x}-1}{7x} = \dfrac{7}{5}\lim\limits_{z\to 0}\dfrac{e^{z}-1}{z} = \dfrac{7}{5}\times 1 = \dfrac{7}{5}$

Example 5: Discuss the existence of $\lim\limits_{x\to 0}\dfrac{1}{x}$

Solution: $\lim\limits_{x\to 0+0}\dfrac{1}{x}=\infty$ where as $\lim\limits_{x\to 0-0}\dfrac{1}{x}=-\infty$

Hence $\lim\limits_{x\to 0}\dfrac{1}{x}$ does not exists.

Example 6: If $f(x) = \dfrac{1}{1+e^{1/x}}\ (x\neq 0)$, prove that $\lim\limits_{x\to 0} f(x)$ does not exists (Cal: 1987)

Solution: Since $x\to 0+0$, $\dfrac{1}{x}\to +\infty$, $e^{1/x}\to\infty$

$\therefore \lim\limits_{x\to 0+0} f(x) = 0$

Again, $x\to 0+0$, $\dfrac{1}{x}\to -\infty$, $e^{1/x}\to 0$

$\therefore \lim\limits_{x\to 0-0} f(x) = 1$

$\therefore \lim\limits_{x\to 0+0} f(x) \neq \lim\limits_{x\to 0-0} f(x)$

Hence $\lim\limits_{x\to 0} f(x)$ does not exists

Example 7: Find $\lim\limits_{h\to 0}\dfrac{\sqrt{x+h}-\sqrt{x}}{h}$, $x>0$

Solution: We first remove the indeterminate form $0/0$ by transforming

$$\frac{\sqrt{x+h}-\sqrt{x}}{h} = \frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \frac{\left(\sqrt{x+h}\right)^2-\left(\sqrt{x}\right)^2}{h\left(\sqrt{x+h}+\sqrt{x}\right)} = \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \frac{1}{\sqrt{x+h}+\sqrt{x}}$$ where by

$$\lim_{h\to 0}\frac{\sqrt{x+h}-\sqrt{x}}{h} = \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Example 8: Show that $\lim_{x\to 0}\frac{\sin\left(x/3\right)}{x} = \frac{1}{3}$

Solution: $\lim_{x\to 0}\frac{\sin\left(x/3\right)}{x} = \lim_{x\to 0}\frac{\sin\left(x/3\right)}{3.x/3} = \frac{1}{3}\lim_{x\to 0}\frac{\sin\left(x/3\right)}{x/3}$

$$= \frac{1}{3}\times 1 = \frac{1}{3}$$

Example 9: Determine the limit L of the function $f(x) = \frac{3x^2+8x-3}{2x+6}$ as $x\to -3$.

Hence find a $\delta > 0$ such that for all x.

$0 < |x-(-3)| < \delta \Rightarrow |f(x) - L| < 0.001$ (NEHU, 2004)

Solution: $\lim_{x\to -3} f(x) = \lim_{x\to -3}\frac{3x^2+8x-3}{2x+6} = \lim_{x\to -3}\frac{(3x-1)(x+3)}{2(x+3)}$

$$= -\frac{10}{2} = L = -5$$

Now for a given $\in = 0.001$ we are to find a $\delta>0$ such $|f(x) - L| < \in$ when $|x-(-3)| < \delta$

i.e $\left|\frac{3x^2+8x-3}{2x+6}+\frac{10}{2}\right| < 0.001$ when $|x+3| < \delta$

i.e $\left|\frac{(3x-1)(x+3)}{2(x+3)}+5\right| < 0.001$ when $|x+3| < \delta$

i.e $|3x-1+10| < 0.001$ when $|x+3| < \delta$

i.e $|3(x+3)| < 0.001$ when $|x+3| < \delta$

i.e $|x+3| < 0.0001/3$ when $|x+3| < \delta$

Hence choosing $\delta = \frac{0.001}{3}$ the conditions are satisfied.

Example 10: Using $\in-\delta$ definition, verify that

$$\lim_{x\to 1}\frac{x^2-1}{x-1}=2 \qquad \text{(NEHU, 2008)}$$

Solution: Let $\in > 0$ be a given arbitrary positive no.

We are to find another no $\delta > 0$ such that

$|f(x) - 2| < \in$ when $|x-1| < \delta$

i.e $\left|\frac{x^2-1}{x-1}-2\right| < \in$ when $|x-1| < \delta$

i.e $\left|\frac{(x-1)(x+1)}{x-1}-2\right| < \in$ when $|x-1| < \delta$

i.e $|x-1| < \in$ when $|x-1| < \delta$

Choosing $\in=\delta$ we see that the condition are statisfied.

Example 11: Using $\in-\delta$ definition, prove that

$$\lim_{x\to 0} x\sin\frac{1}{x}=0 \qquad \text{(NEHU 2009, 2016)}$$

Solution: Let $\in$ be any pre assigned possitive no such that

$|f(x) - 0| < \in$ i.e $\left|x\sin\frac{1}{x}-0\right|<\in$

We are to find a $\delta>0$ corresponding to a given $\in$ such that

$\left|x\sin\frac{1}{x}\right|<\in$ when $|x-0| < \in$ i.e $|x| < \delta$

Since $\left|\sin\frac{1}{x}\right|\leq 1$, by making $|x| < \in$ can make $\left|x\sin\frac{1}{x}\right|<\in$

Hence by choosing $\in=\delta$, the result follows

Example 12: Using $\in - \delta$ definition of limit of a function verify that

$$\lim_{x \to 2} x^2 = 4$$

Solution: Let $\in > 0$ be any pre assigned quantity such that

$$|(f(x) - 4| < \in \text{ i.e } |x^2 - 4| < \in$$

We are to find another positive quantity δ corresponding to a given $\in$ such that $|x^2-4| < \in$ when $|x-2| < \delta$

Since $|x^2-4| < \in \Rightarrow |(x-2)(x+2)| < \in \Rightarrow |x-2| < \in/(x+2)$

Hence by choosing $\delta = \dfrac{\in}{x+2}$, the result follows

Exercises 3.1

1. Establish the following limit by exhibiting δ as a function of $\in$.

(i) $\lim_{x \to 4} 2x = 8$ (ii) $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2} = 4$ (iii) $\lim_{x \to 2} (2x + 1) = 5$

(iv) $\lim_{x \to \infty} \dfrac{1}{x^3} = 0$ (v) $\lim_{x \to 1} \dfrac{1}{(x-1)^2} = \infty$ (vi) $\lim_{x \to \infty} \dfrac{x}{x+1} = 1$

2. Establish the following limits (use definitions only)

(i) $\lim_{x \to 3} \dfrac{1}{x} = \dfrac{1}{3}$ (ii) $\lim_{x \to 3} \dfrac{3}{x} = 1$ (iii) $\lim_{x \to 1} \dfrac{1}{x^2} = 1$

(iv) $\lim_{x \to \infty} \dfrac{x}{1 + x^3} = 0$ (v) $\lim_{x \to 1} \dfrac{x^3 - 1}{x - 1} = 3$

3. Use Cauchy Criterion to discuss the validity of the following statements.

(i) $\lim_{x \to 0} \sin \dfrac{1}{x}$ does not exists

(ii) $\lim_{x \to 0} \cos \dfrac{1}{x}$ does not exists

(iii) $\lim_{x \to 0} \dfrac{1}{1 + e^{1/x}}$ does not exists (Cal 1987)

(iv) $\lim_{x \to 0} x \sin \frac{1}{x} = 0$ (v) $\lim_{x \to 0} x \cos \frac{1}{x} = 0$

(vi) $\lim_{x \to 0} \left(x \sin \frac{1}{x} + \sin \frac{1}{x} + x^2 \sin \frac{1}{x} \right)$ does not exists.

4. In the following examples obtain the right hand and the left hand limits and thus discuss the existence of the limits as x approaches the indicated value of x.

(i) $f(x) = [x], x \to 1$ (ii) $f(x) = \frac{|x|}{x}, x \to 0$

(iii) $f(x) = \frac{\sqrt{1+x^2} - 1}{x}, x \to 0$ (iv) $f(x) = \frac{1}{1+e^{1/x}}, x \to 0$

(v) $f(x) = x$ when $x \leq 1$
$= 2 - x$ when $x > 1, x \to 1$

(vi) $f(x) = \cos x$ when $x \geq 0$
$= -\cos x$ when $x < 0;1\ x \to 0$

(vii) $f(x) = 3$ when x is an integer
$= 0$ when x is not an integer, $x \to 1$

(viii) $f(x) = 4x + 3, x \neq 4$
$= 10, x = 4 ; x \to 4$

Exercise 3.2

1. Evaluate the following limits:

(i) $\lim_{x \to 1} \frac{x^2 + 2x - 2}{2x + 2}$ (ii) $\lim_{x \to 1} \frac{x^2 - 3x + 2}{x^2 - 4x + 3}$

(iii) $\lim_{x \to 0} \frac{a - \sqrt{a^2 - x^2}}{x^2}$ (iv) $\lim_{x \to 0} \frac{(1+x)^2 - 1}{x}$

(v) $\lim_{h \to 0} \frac{1}{h} \left\{ \frac{1}{x+h} - \frac{1}{x} \right\}$ (vi) $\lim_{x \to 0} \frac{x^2}{a - \sqrt{a^2 - x^2}}$

(vii) $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$ (viii) $\lim_{x \to 0} \frac{\frac{1}{x-2} - \frac{1}{x}}{\frac{1}{x-3} + \frac{1}{x}}$

(ix) $\lim_{x\to\infty} \frac{2x^2 - x + 6}{3x^2 + 2x + 1}$ (x) $\lim_{x\to\infty} \frac{1-\sqrt{x}}{1+\sqrt{x}}$

2. Do the following limits exists? If so, find their values

(i) $\lim_{x\to\pi} \frac{1}{\pi - x}$ (ii) $\lim_{x\to\pi} \frac{\sin x}{\pi - x}$ (iii) $\lim_{x\to\pi} \frac{1+\cos x}{\pi - x}$

3. Show that

(i) $\lim_{x\to 0} \frac{\tan x}{x} = 1$ (ii) $\lim_{x\to 0} \frac{3x}{\sin x} = 3$ (iii) $\lim_{x\to 0} \frac{\sin 2x}{\cos x} = 2$

(iv) $\lim_{x\to 0} \frac{1-\cos x}{x} = 0$ (v) $\lim_{x\to 0} \frac{1-\cos 2x}{4x^2} = \frac{1}{2}$ (vi) $\lim_{h\to 0} \frac{\tan ah}{\tan bh} = \frac{a}{b}$

4. Verify the following:

(i) $\lim_{x\to 0} \frac{\sin x^0}{x} = \frac{\pi}{180}$ (ii) $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1$

(iii) $\lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1$ (iv) $\lim_{x\to \pi/4} (\sec 2x - \tan 2x) = 0$

(v) $\lim_{x\to 0} \frac{1-\cos x}{\sin^2 x} = \frac{1}{2}$ (vi) $\lim_{x\to 1} \frac{\sin(1-x)}{1-x^2} = \frac{1}{2}$

5. Given f(x) = |x|, show that $\lim_{h\to 0} \{f(h) - f(0)\}/h$ does not exist.

6. Show that $\lim_{x\to 2} \frac{2x^2 - 8}{x - 2} = 8$. Applying ($\delta$, $\in$) definition, find δ if $\in = 0.1$

7. (i) Is $\lim_{x\to a} \frac{x^2}{x-a} - \lim_{x\to a} \frac{a^2}{x-a} = \lim_{x\to a} \frac{x^2 - a^2}{x-a}$?

(ii) Is $\lim_{x\to a} (x^2 - a^2) \times \lim_{x\to a} \frac{1}{x-a} = \lim_{x\to a} \frac{x^2 - a^2}{x-a}$?

8. Show that:

(i) $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{2x} = e^2$ (ii) $\lim_{x\to\infty} \left(1 + \frac{2}{x}\right)^{3x} = e^6$

(iii) $\lim_{x \to 0} \frac{e^{4x} - 1}{x} = 4$ (iv) $\lim_{x \to a} \frac{x^8 - a^8}{x^3 - a^3} = \frac{8}{3} a^5$

(v) $\lim_{x \to 0} (1 + 2x)^{\frac{x+3}{x}} = e^6$ (vi) $\lim_{x \to 0} \frac{e^{-x} - 1}{x} = -1$

9. Prove that $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$ and apply it to show that

$$\lim_{x \to 0} \frac{\log(1+8x)}{x} = 8 \text{ and } \lim_{x \to 0} \frac{\log(1+8x)}{(1+7x)} = \frac{8}{7}$$

10. Show that $\lim_{x \to 0} \frac{\sin x}{\sin x} = 1$ but $\lim_{x \to 0} \frac{\sin\left(\frac{1}{x}\right)}{\sin\left(\frac{1}{x}\right)}$ does not exists.

4

Continuity

Introduction

In the definition of the limit of a function f(x) as $x \to a$ does not require f(a) to exist. This restriction enables us to apply the definition to function that fail to have meaning at x = a, but have definite values for all other values of x near the point x = a.

The class of functions in which limits can be found by this substitution process are called continuous functions. Thus continuity means identity of limits with values.

Definitions

4.1 Continuity at a point

A real valued function f(x) is said to be continuous at x=a, provided $\lim_{x \to a} f(x)$ exists, is finite and is equal to f(a)

In otherwords, for f(x) to be continuous at x=a, the following conditions are to be met

(i) f(a) is defined

(ii) $\lim_{x \to a} f(x)$ exists

(iii) $\lim_{x \to a} f(x) = f(a)$

Thus, a function f(x) is said to be continuous at x=a, if corresponding to any pre-assigned positive quantity $\in$ however small it may be, there exists a positive number δ, such that $|f(x) - f(a)| < \in$ for $|x-a| < \delta$.

If f(x) is continuous for every value of x in the interval (a, b), it is said to be continuous throughout the interval.

If f(x) is not continuous at x=a, it is said to have a discontinuity at that point and this point x=a is called a point of discontinuity.

4.2 Different Classes of Discontinuity

(A) If $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$, then f(x) is said to have an ordinary discontinuity (or discontinuity of the first kind) at x=a. In this case, f(a) may or may not exists, or if it exists, it may be equal to one of $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ or may be equal to neither.

(B) If $\lim_{x \to a^-} f(x) = \lim_{x \to a^-} f(x) \neq f(a)$, or is not defined, then f(x) is said to have a removable discontinuity at x = a.

In this case, the funcion can be made continuous there by suitably defining the function at the particular point only.

These two classes of discontinuties (A) and (B) are termed simple discontinuity.

(C) If one or both of $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^-} f(x)$ tend to $+\infty$ or $-\infty$ then f(x) is said to have an infinite discontinuity at a. Here f(a) may or may not exist.

(D) Any point of discontinuty which is not a point of simple discontinuity, nor an infinity is called a point of oscillatory discontinuity. At such a point the function may oscillate finitely or infinitely and does not tend to a limit, or tends to $+\infty$ or $-\infty$

Examples:

1. $f(x) = \left(2 + e^{\frac{1}{x}}\right)^{-1}$ has an ordinary discontinuity at x = 0 since $\lim_{x \to 0^+} f(x) = 0$ and $\lim_{x \to 0^-} f(x) = \frac{1}{2}$

2. $f(x) = \frac{x^2 - a^2}{x - a}$ has a removable discontiunity at x = a since f(a) is undefined here, though $\lim_{x \to a} f(x)$ exists and = 2a

3. $f(x) = e^{-\frac{1}{x-a}}$ has an infinite discontinuity at x = a since $\lim_{x \to a^-} f(x) \to \infty$

4. $f(x) = \sin\frac{1}{x}$ oscillates finitely at x = 0 since the point x = 0 is an oscillatory discontinuity of $\sin\frac{1}{x}$ and $\sin\frac{1}{x}$ assumes every values between –1 and +1

4.3 Some Properties of Continuous Functions

(i) The sum or difference of two continuous functions is a continuous function.

i.e if f(x) and g(x) are both continuous at x=a, then f(x) ± g(x) is continuous at x=a

Proof: By definition of continuity $\lim_{x \to a} f(x)$ exists and = f(a) and $\lim_{x \to a} g(x)$ exists and = g(a)

Hence $\lim_{x \to a} \{f(x) \pm g(x)\} = \lim_{x \to a} \{f(x) \pm g(x)\}$

$$= f(a) \pm g(a)$$

where, bydefinition f(x) ± g(x) is continuous at x=a

Note 1. The result may be extended to the case if any finite number of functions.

Note 2. If f(x) is continuous at x=a and g(x) is not, then f(x) ± g(x) is discontinuous at x=a and behaves like g(x).

(ii) The product of two continuous functions is a continuous function.

i.e if f(x) and g(x) are both continuous at x=a, then f(x).g(x) is also continuous at x=a

Proof: Exactly similar to the above case

Note: The result may be extended to any finite number of functions.

(iii) The quotient of two continuous functions is a continuous function, provided the denominator is not zero anywhere for the range of values considered.

i.e if f(x) and g(x) are both continuous at x=a and g(a)≠0, then $\frac{f(x)}{g(x)}$ is continuous at x=a

Proof: Same as in the above case

(iv) If f(x) is continuous at x=a and $f(a)\neq 0$, then in the neighbourhood of x=a, f(x) has the same sign as that of f(a) i.e we can get a positive quantity δ, such that f(x) preserves the same sign as that of f(a) for every value of f(x) is the interval $a-\delta < x < a+\delta$

Proof: Since f(x) is continuous at x=a, then by definition, to an arbitrary chosen positive number $\in$, we can get a positive number δ such that

$|f(x) - f(a)| < \in$ for $|x - a| < \delta$

i.e $f(a) - \in < f(x) < f(a) + \in$ for $a-\delta < x < a+\delta$

As $f(a)\neq 0$, if $f(a) > 0$, choose $\in = \frac{1}{2}$ f(a), then from above $f(x) > f(a) - \in$

i.e $f(x) > \frac{1}{2}$ f(a) and is accordingly positive when $a-\delta < x < a+\delta$

If $f(a) < 0$, choose $\in = -\frac{1}{2}$ f(a), then from above $f(x) < f(a) + \in$

i.e $f(x) < f(a) - \frac{1}{2}$ f(a) i.e $f(x) < \frac{1}{2}$ f(a) and is accordingly negative when $a-\delta < x < a+\delta$

Thus, whatever be the sign of f(a), we can find δ such that f(x) has the same sign as thatg of f(a) in the range $a-\delta < x < a+\delta$

(v) If f(x) is continuous throughout the internal (a, b) and if f(a) and f(b) are of oppositive signs, then there is at least one value of x say $\xi \in (a, b)$ such that $f(\xi) = 0$.

Proof: Proof is beyond the scope of this book

(vi) If f(x) is continuous throughout the interval (a, b) and if $f(a)\neq f(b)$, then f(x) assumes every value between f(a) and f(b) at least once in the interval

Proof: Let k be any quantity such that $f(a) < k < f(b)$

Consider $\phi(x) = f(x) - k$ ....(i)

Since f(x) is continuous in the interval (a, b), $\phi(x)$ is also continuous there

Also $\phi(a) = f(a)-k < 0$ and $\phi(b) = f(b)-k > 0$

i.e $\phi(a)$ and $\phi(b)$ are of opposite signs, hence by (v) above there is at least one value of $x = \xi$ in (a, b) such that $f(\xi)=0$ i.e $f(\xi)-k=0$ i.e $f(\xi) = k$

Hence in otherwords, f(x) assumes the value k at some point in the interval, between f(a) and f(b)

(vii) A function which is continuous throughout a closed interval [a, b] is bounded therein

(viii) A function continuous in a closed interval [a, b] attains it lower and upper bounds, at least once each in the interval

Proof: Since f(x) is continous in the closed interval [a, b], hence by (vii) it is bounded there in.

Let M and m be the upper and lower bounds of f(x) in [a, b] we are to prove that f(x) attains the value M and m at least once

If possible, suppose f(x) does not attain the value M in [a, b] i.e $f(x) \neq M$ $\forall x \in [a. b]$

Consider the function $\phi(x) = \frac{1}{M - f(x)}$, $x \in [a. b]$

Since $f(x) \neq M \ \forall x \in [a. b]$, $\phi(x)$ is continuous in [a, b]

Since M is the upper bound of f(x) in [a, b]

$\therefore f(x) > M - k$ where $k>0$

i.e $k > M - f(x)$ i.e $\frac{1}{M - f(x)} > \frac{1}{k}$ i.e $\phi(x) > k$

Contradicting that $\phi(x)$ is bounded

Henc our assuption that $f(x) \neq M \ \forall x \in [a. b]$ is wrong and therefore f(x) attains the value M at least once in [a, b]

Similarly we can prove that f(x) attains the value m at least once in [a, b]

Hence f(x) attains its lower and upper bounds at least once in the internal.

4.4 Continuity of Some Elementary Functions

(i) Function x^n. Where n is any rational number.

We know $\lim_{x \to a} x^n = a^n$ for all values of n except when a = 0 and n is negative

Hence x^n is continuous for all values of x when n is positive and continuous for all values of x except o when n is negative

When n is negative i.e $n = -m$, $m>0$ $x^n = x^{-m} = \frac{1}{x^m}$ which either does not tend to a limit or $\to \infty$ as $x \to 0$

(ii) Polynomials

Any polynomial $P(x) = a_0x^n + a_1x^{n-1} + ... + a^n$ is the sum of finite number of positive integral powers of x (each multiplied by a constant) each of which is continuous for all values of x, the polynomials itself is continuous for all values of x.

(iii) Rational Algebraic Function

Rational algebraic functions $\frac{a_0x^n + a_1x^{n-1} + ... + a_n}{b_0x^n + b_1x^{n-1} + ... + b_n} = \frac{Q(x)}{R(x)}$ being quotient of two polynomials, are continuous for all values of x except those which make the denominator zero.

Illustrative Examples

Example 1. A function f(x) is defined as follows:

$f(x) = x$ when $x>0$, $f(0) = 0$, $f(x) = -x$ when $x<0$

Prove that the function is continuous at x=0

Solution: Here $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-x) = 0$

Thus $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) = f(0) = 0$

Hence f(x) is continuous at x = 0

Example 2: A function f(x) is defined as follows

$$f(x) = x \sin \frac{1}{x} \text{ for } x \neq 0$$

$$= 0 \text{ for } x=0$$

Show that f(x) is continuous at x=0

Solution: Let $\in$ be any assigned positive quantity, however small such that

$|f(x) - f(0) < \in$ i.e $|x \sin \frac{1}{x} - 0| < \in$

We are to find a $\delta>0$ corresponding to the above given $\in$ such that

$|x \sin \frac{1}{x}| < \in$ when $|x-0\} < \delta$ i.e $|x| < \delta$

Since $|\sin \frac{1}{x}| \leq 1$ by making $|x| < \in$ we can make $|x \cdot \sin \frac{1}{x}| < \in$

Hence by choosing $\delta = \in$ we see that $\lim_{x\to 0} f(x) = \lim_{x\to 0} x \sin \frac{1}{x} = 0 = f(0)$

$\therefore$ f(x) is continuous at x=0

Example 3: Prove that the function $f(x) = e^x$ is continuous for every value of x

Solution: Corresponding to any preassigned positive number $\in$, however small, we can choose n sufficiently large such that $(1+\in)^n > e$

[Since $((1+\in)^n > 1+n\in$, e is finite]

Thus $e^{1/n} - 1 < \in$

Hence if $0 < x < \dfrac{1}{n}$, $e^x - 1 < e^{1/n} - 1 < \in$

Therefore $\lim\limits_{x \to 0^+} (e^x - 1) = 0$ or $\lim\limits_{x \to 0^+} e^x = 1$

If x is negative, putting $x = -y$, $y>0$

$$\lim_{x \to 0^-} e^x = \lim_{y \to 0^+} \frac{1}{e^y} = 1$$

Hence $\lim\limits_{x \to 0} e^x = 1$

$\therefore \lim\limits_{x \to c} e^{x-c} = 1$ i.e $\lim\limits_{x \to c} e^x = e^c$

$\therefore$ e^x is continuous at any point x=c.

Since x=c is arbitrary, $f(x) = e^x$ is continuous for every value of x

Example 4: The function $f(x) = \log x$, $x>0$ is continuous

Solution: Note that logx is defined only for values of $x>0$

Let $\log x = y$ and $\log(x+h) = y+k$

Then $e^y = x$ and $e^{y+k} = x+h$

$\therefore h = e^{y+k} - e^y$

Since e^y is continuous (by example 3) function of y, $e^{y+k} \to e^y$ i.e $h \to 0$ as $k \to 0$

Hence $\{\log(x+h) - \log x\} \to 0$ as $k \to 0$ i.e as $h \to 0$

Therefore logx is continuous

Example 5: If $f(x) = \dfrac{3x^3 + 5x^2 + 7x}{\sin x}$, $x \neq 0$ is tobe continuous at x=0. Show that f(0) must be equal to 7.

Solution: $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \dfrac{3x^3 + 5x^2 + 7x}{\sin x}$

$$= \lim_{x \to 0} \frac{x}{\sin x} \times (3x^2 + 5x + 7)$$

$$= 7$$

Therefore for contiuity of f(x) at x=0, f(0) must be equal of 7

Example 6: Using $\in - \delta$ definition of continuity, prove that y=sinx is continuous at every value of x. (NEHU, 2001)

Solution: Let $\in$ be any pre assigned positive number, however small. We have to show that there exists a positive number δ corresponding to $\in$ such that

$|\sin x - \sin a| < \in$ when $|x-a| < \delta$

Since $|\sin x - \sin a| = \left|2\cos\frac{x+a}{2}\sin\frac{x-a}{2}\right|$

Now $\left|\cos\frac{x+a}{2}\right| \leq 1$ and $\left|\sin\frac{x-a}{2}\right| \leq \left|\frac{x-a}{2}\right|$

Hence $|\sin x - \sin a| \leq |xa| < \in$

Choosing $\delta = \in$ we see that $|\sin x - \sin a| < \in$ when $|x-a| < \delta$

i.e $\lim_{x \to a} \sin x = \sin a$

$\therefore$ y = sinx is continuous at x=a

Since x=a is arbitrary, y=sinx is continuous for every value of x.

Example 7: Consider the function (NEHU, 2004)

$$f(x) = \begin{cases} \frac{x^2-16}{x-4}, & x \neq 4 \\ 0 & , x = 4 \end{cases}$$

Is it continuous at x = 4? Is it possible to make f continuous at x=4 by redefining the value of f(4)? Answer with justification. (NEHU, 2004)

Solution: $\lim_{x \to 4} f(x) = \lim_{x \to 4} \frac{x^2-16}{x-4} = \lim_{x \to 4} \frac{(x-4)(x+4)}{x-4}$

$$= \lim_{x \to 4} (x + 4) = 8$$

Since $\lim_{x \to 4} f(x) = 8 \neq f(4) = 0$

$\therefore$ f(x) is not continuous at x=4.

Here x=4 is a point removable discontinuity.

f(x) can be made continuous at x=4 by redefining $f(4) = 8 = \lim_{x\to 4} f(x)$

Hence for f(x) to be continuous at x=4, we must have

$$f(x) = \begin{cases} \dfrac{x^2-16}{x-4}, & x \neq 4 \\ 8 & , x = 4 \end{cases}$$

Example 8: Using $\in-\delta$ definition of continuity, prove that y=cosx is continuous in IR. (NEHU, 2005)

Solution: Same as example 6.

Example 9: Discuss the right continuity, the left continuity and then the continuity of the function

$$f(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & \text{for } x < 1 \\ 2 & \text{for } x = 1 \\ x+2, & \text{for } x > 1 \end{cases}$$

at the point x=1 (NEHU, 2006, 2015)

Solution: $\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1+h) = \lim_{h\to 0}(1+h) + 2 = 3 \neq 2 = f(1)$

f(x) is not right continuous at x=1

Again, $\lim_{x\to 1^-} f(x) = \lim_{h\to 0} f(1-h) = \lim_{h\to 0} \dfrac{(1-h)^2-1}{1-h-1}$

$$= \lim_{h\to 0} \frac{1+h^2-2h-1}{1-h-1}$$

$$= \lim_{h\to 0} \frac{h(h-2)}{-h} = 2 = f(1)$$

$\therefore$ f(x) is left continuous at x=1

Since $\lim_{x\to 1^+} f(x) \neq \lim_{x\to 1^-} f(x) = f(2)$

$\therefore$ f(x) is not continuous at x=1

Example 10: What value should be assigned to a to make the function

$$f(x) = \begin{cases} x^2 - 1 & , x < 3 \\ 2ax & , x \geq 3 \end{cases}$$

continuous at x=3? Justify your answer. (NEHU, 2006)

Solution: If f(x) is continuous at x=3, then

$$\lim_{x \to 3} f(x) = f(3)$$

i.e $\lim_{x \to 3} x^2 - 1 = 6a$

i.e $8 = 6a \Rightarrow a = \frac{8}{6} = \frac{4}{3}$

Example 11: If $f(x) = \begin{cases} x^3 \cos\frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases}$

use $\in - \delta$ definition show that f(x) is continuous at x=0 (NEHU, 2007)

Solution: Let $\in > \delta$ be an arbitrary given number, we can find a number $\delta > 0$ coresponding to this given $\in$ such that

$$|f(x) - f(0)| < \in \text{ when } |x - 0| < \delta$$

i.e $\left| x^3 \cos\frac{1}{x} - 0 \right| < \in$ when $|x| < \delta$

i.e $\left| x^3 \cos\frac{1}{x} \right| < \in$ when $|x| < \delta$

Since $\left| \cos\frac{1}{x} \right| \leq \in$ by making $|x| < \in^{\frac{1}{3}}$ we see that the condition is satisfied.

$$\therefore f(x) = \begin{cases} x^3 \cos\frac{1}{x} & , x \neq 0 \\ 0 & , x = 0 \end{cases} \text{ is continuous at } x=0$$

Example 12: For what value of a the function defined by f(x) = 2ax+3 when $x \neq 2$ and f(x) = 23 when x=2 is continuous at x=2? (NEHU, 2009)

Solution: For f(x) to be continuous at x=2, we must have

$$\lim_{x\to 2} f(x) = f(2)$$

i.e $\lim_{x\to 2} (2ax+3) = 23$

i.e $4a+3 = 23 \Rightarrow 4a = 20 \Rightarrow a=5$

Example 13: Consider the function

$$f(x) = \begin{cases} 3x+2, & x<0 \\ x+1, & x\geq 0 \end{cases}$$

Is this function continuous at x=0? (NEHU, 2010, 2016)

Solution: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (3x+2) = 2$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x+1) = 1$$

Since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

$\therefore$ f(x) is not continuous at x=0

Example 14: A function defined as

$$f(x) = x^2 \cos\frac{1}{x}, \text{ when } x\neq 0$$

$$= 0, \text{ when } x=0$$

Is f(x) continuous at x=0? (Use $\in-\delta$ definition to justify your answer)
(NEHU, 2013)

Solution: Same as example 11.

Example 15: A function f is defined as

$$f(x) = \begin{cases} x^2+2x+b, & x\neq 0 \\ -3, & x=0 \end{cases}$$

For what value of b is the function continuous at x=0? (NEHU, 2016)

Solution: f(x) is continuous at x=0 if

$$\lim_{x\to 0} f(x) = f(0)$$

i.e $\lim_{x\to 0} (x^2 + 2x + b) = -3$

i.e $b = -3$

Exercise

1. A function f(x) is defined as follows

$f(x) = x^2$, when $x \neq 1$

$= 2$, when $x=1$

Is f(x) continuous at x=1?

2. A function f(x) is defined as follows

$f(x) = \frac{1}{2} - x$ when $0 < x < \frac{1}{2}$

$= \frac{1}{2}$ when $x = \frac{1}{2}$

$= \frac{3}{2} - x$ when $\frac{1}{2} < x < 1$

Show that f(x) is discontinuous at $x = \frac{1}{2}$

3. A function $\phi(x)$ is defined as follows

$\phi(x) = x^2$ when $x<1$

$= 2.5$ when $x=1$

$= x^2+2$ when $x>1$

Is $\phi(x)$ continuous at x = 1?

4. A function f(x) is defined in the following way

$f(x) = -x$ when $x \leq 0$

$= x$ when $0<x<1$

$= 2-x$ when $x \geq 1$

Show that it is continuous at x=0 and x=1

5. The function $f(x) = \frac{x^2-16}{x-4}$ is undefined at x=4, what value must be assigned to f(4), if f(x) is to be continuous at x=4?

6. Determine whether the following functions are continuous at x= 0

(i) $f(x) = \frac{x^4+x^3+2x^2}{\sin x}$; f(0) = 0 (ii) $f(x) = \frac{x^4+4x^3+2x}{\sin x}$; f(0) = 0

7. Find the points of discontiuity of the following functions:

(i) $\dfrac{x^2+2x+5}{x^2-8x+12}$ (ii) $\dfrac{x^3+2x+5}{x^2-8x+16}$

8. The function f(x) is defined as follows

$f(x) = 0$, when $x^2>1$

$= 1$, when $x^2<1$

$= \frac{1}{2}$ when $x^2=1$

Draw the diagram of the function and discuss from the diagram that, except at the point x=1 and x=–1 the function is contiunuous. Discuss also why the function is discontinuous at these two points although it has a value for every value of x.

4.5 Uniform Continuity

A function f(x) is said to be uniformly continuous in an interval if to a given $\epsilon(>0)$ however small, there exists a positive number δ, independent of a such that when $|x-a|<\delta$, $|f(x) - f(a)| < \epsilon$

Example 1: Consider $f(x) = x^2$, x in [–a, a] (NEHU, 2000)

Solution: Let x^1 be any point in [–a, a] then

$|f(x) - f(x^1)| = |x^2 - x^{12}| = |x - x^1|\,|x + x^1| \leq 2a|x-x^1|$

i.e we have $|f(x) - f(x^1)| < \epsilon$ whenever $|x-x^1| < \epsilon/2a$

By choosing $\delta = \epsilon/2a$ (independent of x^1) we see that the condition is satisfied and $f(x) = x^2$ is uniformly continuous in the given interval.

Example 2: Show that $f(x) = x^2$ is not uniformly continuous in IR

Solution: Let $\epsilon>0$ be given (however small) and $x^1>0$ such that x^1 is less than any fixed positive number M. Then

$|f(x) - f(x^1)| = |x^2 - x^{12}| = |x - x^1|\,|x + x^1| < 2M\,|x - x^1|$

Hence, $|f(x) - f(x^1)| < \epsilon$ if $|x - x^1| < \epsilon/2M$

But if we choose $\delta = \epsilon/2M$, we see that δ is dependent on M and hence on x^1

Therefore $f(x) = x^2$ is not uniformly continuous on IR

Theorem 1: A uniformly continuous function in an interval is necessarily continuous in that interval

Proof: Follows from definition of uniform continuity.

Theorem 2: A function which is contnuous in a closed interval is also uniformly continuous in that interval

Proof: Follows from definition.

Exercise

1. Define uniform continuity of a function in an interval show that the function $f(x) = x^2$ is uniformly continuous in $[-2, 2]$ (NEHU, 2005)

   (Solution: same as example 1)

2. Let $f(x) = x^2$ be a function defined in $[-a, a]$. Show that f is uniformly continuous in $[-a, a]$ but not in $(-\infty, \infty)$ (NEHU, 2009)

   (Solution: same as examples 1, 2)

3. Show that $f(x) = \frac{1}{x}$ is uniformly continuous in the interval $[1, 2]$

   (NEHU, 2007)

5

Derivative or Differentiation

Introduction

Differentation is about finding the rates of change of one quantity compared to another i.e when the rate of change is not constant. In other words derivative is the instantaneous rate of change of a function with respect to one of its vanables. Denvative or Differentation is equivalent to finding the slope of the tangent line to the function at a point. In fact the essense of calculus is the derivative.

Definitions

5.1 Increment

The increment of a vanable in changing from one value to another is the difference obtained by subtracting the initial value from its findal value. This increment may be positive or negative according as the vanable in changing increases or decreases and is denoted by any one of the symbols h, Δx (delta x) or δx (delta x)

If in $y = f(x)$, the independent variable x takes an increament Δx (or h), then Δy (or k) is the corresponding increment of y.

i,e $y+\Delta y = f(x+\Delta x)$ i.e. $\Delta y = f(x+\Delta x) - f(x)$

or $y+k = f(x+h)$ i.e. $k = f(x+h) - f(x)$

5.2 Differential Coefficient (or Derivative)

Let $y = f(x)$ be a finite, single valued function defined in any interval of x and assume x to have particular value in the interval. Let Δx (or h) be the increment of x and Δy (or k) by the corresponding increment of y. If the ratio $\Delta y/\Delta x$ of these increments tends to a definite finite limit as Δx tends to zero, then this limit is called *the differential coefficient (or derivative) of f(x) (or y)* for the particular value of x and is denoted by $f'(x)$ or $\frac{d}{dx}\{f(x)\}$ or $\frac{dy}{dx}$ or D $\{f(x)\}$

Thus symbolically, the differential coefficient of y = f(x) with respect to x is

$$f'(x) \text{ or } \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$\text{or } f'(x) \text{ or } \frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

Remark: If $\frac{\Delta y}{\Delta x} \to \infty$ or $-\infty$ as $\Delta x \to 0$, then also we say that the derivative exists and $= +\infty$ or ∞.

Note: The process of finding the differential coefficient is called differentiation and we are said to differentiate f(x) or to differentiate f(x) with respect to x, to emphasis that x is an independent variable.

5.3 Let y=f(x) be a finite real valued function defined in an interval containing an interor point c. Then

$$\lim_{x \to c} \frac{f(x)-f(c)}{x-c} \text{ or } \lim_{h \to 0} \frac{f(c+h)-f(c)}{h}$$ if it exists, is called the derivative

of f(x) at x=c, denoted by $f'(c)$ or dy/dx at x=c. If $f'(c)$ exists, we say that f is *derivable or differentiable* at x=c

The number $\lim_{x \to c+0} \frac{f(x)-f(c)}{x-c}$ or $\lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h}$ if it exists is called the right-hand derivative of f(x) at x=c denoted by $Rf'(c)$

Similarly, the number $\lim_{h \to c-0} \frac{f(x)-f(c)}{x-c}$ or $\lim_{h \to 0^-} \frac{f(c+h)-f(c)}{h}$ if it exists, is called the left-hand derivative of f(x) at x=c denoted by $L\,f'(c)$

Thus for $f'(c)$ to exists, $R\,f'(c)$ must be equal to $L\,f'(c)$. If either one fails to exists or both exists and have different values, then $f'(c)$ does not exists.

5.4 Continuity of a Derivable Function

Theorem: If a function has a finite derivative at a given point, it is continuous at this point.

or

If $f'(a)$ is finite, $f(x)$ must be continuous at $x=a$

Proof: $f'(a) = \lim_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ exists

$$\therefore \lim_{h\to 0} \{f(a+h) - f(a)\} = \lim_{h\to 0} \left\{\frac{f(a+h)-f(a)}{h} \times h\right\}$$

$$= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \times \lim_{h\to 0} h$$

(since both limits exists)

$$= f'(a)\times 0 = 0 \text{ (since } f'(a) \text{ is finite)}$$

$$\therefore \lim_{h\to 0} f(a+h) = f(a)$$

$\therefore$ $f(x)$ is continuous at $x=a$

Remark: The converse of the theorem is not necessarily true, that is a function may be continuous at a point yet not have a derivative at that point.

For example, consider the function $f(x) = |x|$

Then $f(x) = |x|$ is continuous at $x=0$ since for continuity

$|f(x) - f(0)| = |x| < \epsilon$

Whenever $|x-0| = |x| < \delta$ and choosing $\delta = \epsilon$, then conditions of continuity are satisfied.

But $\dfrac{f(0+h)-f(0)}{h} = \dfrac{f(h)}{h} = \dfrac{|h|}{h} \begin{cases} 1 & , h>0 \\ -1 & , h<0 \end{cases}$ and hence $Rf'(0)=1$ but $Lf'(0) = -1$

So $f'(0)$ does not exists.

Corollary: If $f(x)$ is discontinuous at $x=c$, then it can not have a derivative at that point.

5.5 Differential Coefficients in some standard cases

(1) Differntial Coefficient of x^n

Let $f(x) = x^n$

Then from definition

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

Putting X = x+h i.e. h = X–x then when $h\to 0$, $X\to x$

$$\text{Hence } f'(x) = \lim_{X\to x} \frac{X^n - x^n}{X - x} = nx^{n-1} \text{ for all values of n}$$

$$\text{Thus } \frac{d}{dx} f(x) = \frac{d}{dx} x^n = f'(x) = n\, x^{n-1}$$

$$\text{Cor: } \frac{d}{dx}\, x = 1;\ \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}};\ \frac{d}{dx}\, x^{-n} = \frac{n}{x^{n+1}}$$

(ii) Differential Coefficient of e^x

Let $f(x) = e^x$

Then from definition

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{e^{x-h} - e^x}{h}$$

$$= \lim_{h\to 0} e^x . \frac{e^h - 1}{h}$$

$$= e^x \left[\because \lim_{h\to 0} \frac{e^h - 1}{h} = 1\right]$$

$$\text{Thus } \frac{d}{dx}\, e^x = e^x$$

(iii) Differential Coefficient of a^x

Let $f(x) = a^x$

Then from definition

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h}$$

$$= a^x \lim_{h\to 0} \frac{a^h - 1}{h}$$

$$= a^x \log a \left[\because \lim_{h\to 0} \frac{a^h - 1}{h} = \log a\right]$$

$$\text{Thus } \frac{d}{dx}\, a^x = a^x \log a$$

(iv) Differential Coefficient of logx

Let f(x) = logx

Then from definition

$$f^1(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\log(x+h)-\log x}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \log\left(\frac{x+h}{x}\right)$$

$$= \lim_{h\to 0} \frac{1}{x}.\frac{x}{h} \log\left(1+\frac{h}{x}\right)$$

$$= \frac{1}{x} \lim_{Z\to 0} \frac{1}{z} \log(1+z) \text{ where } z = \frac{h}{x}$$

$$= \frac{1}{x} \; [\because \lim_{z\to 0} \log\,(1+z)^{\frac{1}{z}} = 1]$$

Thus $\frac{d}{dx} \log x = \frac{1}{x}$

Cor: $\frac{d}{dx} \log_a^x = \frac{1}{x} \log_a^e$

(v) Differential Coefficient of Sinx

Let f(x) = Sinx

Then from definition

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\mathrm{Sin}(x+h)-\mathrm{Sin}x}{h}$$

$$= \lim_{h\to 0} \frac{2\,\mathrm{Sin}\left(\frac{x+h-x}{2}\right)\mathrm{Cos}\left(\frac{x+h+x}{2}\right)}{h}$$

$$= \lim_{h\to 0} \frac{2\,\mathrm{Sin}\frac{h}{2}\,\mathrm{Cos}\left(x+\frac{h}{2}\right)}{h}$$

$$= \lim_{h\to 0} \frac{\text{Sin}\frac{h}{2}}{\frac{h}{2}} \text{Cos}\left(x+\frac{h}{2}\right)$$

$$= \text{Cosx}$$

[Since as $h \to 0$, Cosx being continuous function of x, $\text{Cos}\left(x+\frac{h}{2}\right) \to \text{Cosx}$

and also $\lim_{h\to 0} \frac{\text{Sin}\frac{h}{2}}{\frac{h}{2}} = 1$]

Thus $\frac{d}{dx}\text{Sinx} = \text{Cosx}$

(vi) Differential Coefficient of Cosx

Let f(x) = Cosx

Then from definition

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\text{Cos}(x+h)-\text{Cosx}}{h}$$

$$= \lim_{h\to 0} \frac{-2\sin{}^h\!/_2 \sin\left(x+{}^h\!/_2\right)}{h}$$

$$= -\lim_{h\to 0} \frac{\sin{}^h\!/_2}{{}^h\!/_2} \text{sinx}\left(x+{}^h\!/_2\right)$$

$$= -\text{sinx as in (iv)}$$

(vii) Differential Coefficiant of tanx

Let f(x) = tanx

Then from definition

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\tan(x+h)-\tan x}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x}\right]$$

$$= \lim_{h \to 0} \left[\frac{\cos x \sin(x+h) - \sin x \cos x(x+h)}{h \cos(x+h) \cos x} \right]$$

$$= \lim_{h \to 0} \frac{\sin(x+h-x)}{h \cos(x+h) \cos x}$$

$$= \lim_{h \to 0} \frac{\sin h}{h} \cdot \frac{1}{\cos(x+h)\cos x}$$

$$= \frac{1}{\cos^2 x} \quad \left[x \neq \frac{1}{2}(2n+1)\pi \right]$$

Siince $\lim_{h \to 0} \frac{\sin h}{h} = 1$ and $\lim_{h \to 0} \cos(x+h) = \cos x$

Thus $\frac{d}{dx} \tan x = \sec^2 x \quad \left[x \neq \frac{1}{2}(2n+1)\pi \right]$

(viii) Differential Coefficient of cotx

Similar proof as in (vii) above

(ix) Differential Coefficient of secx

Let f(x) = secx

Then from definition

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sec(x+h) - \sec x}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \left[\frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right]$$

$$= \lim_{h \to 0} \frac{\cos x - \cos(x+h)}{h \cos(x+h) \cos x}$$

$$= \lim_{h \to 0} \frac{2 \sin h/2 \sin\left(x + h/2\right)}{h \cos(x+h) \cos x}$$

$$= \lim_{h \to 0} \frac{\sin h/2}{h/2} \cdot \sin\left(x + h/2\right) \cdot \frac{1}{\cos(x+h) \cos zx}$$

$$= 1 . \sin x . \frac{1}{\cos^2 x} = \tan x \sec x$$

Since $\lim_{h\to 0} \frac{\sin h/2}{h/2} = 1, \quad \lim_{h\to 0} \sin\left(x + h/2\right) = \sin x$

and $\lim_{h\to 0} \cos(x+h) = \cos x$

Thus $\frac{d}{dx} \sec x = \sec x \tan x$

(x) Differential Coefficientg of cosec x

Similar proof as in (ix) above

5.6 Fundamental Theorems on Differentiation

(In following theorems, the functions f(x), g(x) are assumed to be continuous and f′(x), g′(x) exist)

Theorem 1: The differential cofficient of a constant function is zero.

Proof: Let f(x) = k, where k is a constant for every value of x

Then f(x+h) = k

Now $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{k-k}{h} = 0$

i.e $\frac{d}{dx}\{(fx)\} = 0$ i.e $\frac{d}{dx}(k) = 0$

Theorem 2: The differential coefficient of the product of the constant and a function is equal to the product of the constant and the differential coefficient of the function.

i.e $\frac{d}{dx}\{k.f(x)\} = k \frac{d}{dx}\{(fx)\} = k\, f'(x)$ where k is a constant

Proof: $\frac{d}{dx}\{k.f(x)\} = \lim_{h\to 0} \frac{kf(x+h)-kf(x)}{h}$

$$= k \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= k\, f'(x) = k \frac{d}{dx}\{f(x)\}$$

Theorem 3: The differential coefficient of the sum (or difference) of two functions is the sum (or the difference) of their differential coefficients

$$\text{i.e } \frac{d}{dx}\{f(x) \pm g(x)\} = \frac{d}{dx}\{f(x) \pm g(x)\}$$

Proof: $\frac{d}{dx}\{f(x) \pm g(x)\} = \lim_{h\to 0} \frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}$

$$= \lim_{h\to 0} \frac{[f(x+h)-f(x)]+[g(x+h)-g(x)]}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$$

$$= \frac{d}{dx}\{f(x)\} + \frac{d}{dx}\{g(x)\}$$

$$= f'(x) + g'(x)$$

Similarly, $\frac{d}{dx}\{f(x)\} - \{g(x)\} = f'(x) - g'(x)$

Theorem 4: The differential coefficient of the product of two function = first function $\times$ differential coefficient of the second function + second function $\times$ differential coefficient of the first function.

$$\text{i.e. } \frac{d}{dx}\{f(x)\}.\{g(x)\} = f(x).\frac{d}{dx}\{g(x)\} + g(x).\frac{d}{dx}\{(fx)\}$$

$$= f(x)\, g'(x) + g(x)\, f'(x)$$

Proof: $\frac{d}{dx}\{f(x)\}.\{g(x)\} = \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)-f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \left[f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h}\right]$$

$$= \lim_{h\to 0} f(x+h)\, \frac{g(x+h)-g(x)}{h} + \lim_{h\to 0} f(x)\, \frac{f(x+h)-f(x)}{h}$$

$$= f(x)\ \frac{d}{dx}\{g(x)\} + g(x)\frac{d}{dx}\{f(x)\}$$

$$= f(x)\ g'(x) + g(x)\ f'(x)$$

Theorem 5: The differential coefficient of the quotient of two functions

$$= \frac{(\text{differential coefficient of numerator}).\text{denominator} - (\text{differential coefficient of denominator}).\text{numerator}}{\text{Square of denominator}}$$

i.e. $\dfrac{d}{dx}\left\{\dfrac{f(x)}{g(x)}\right\} = \dfrac{f'(x)g(x) - g'(x)f(x)}{\{g(x)\}^2}$ provided $g(x) \neq 0$

Proof: $\dfrac{d}{dx}\left\{\dfrac{f(x)}{g(x)}\right\} = \lim\limits_{h\to 0} \dfrac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h}$

$$= \lim_{h\to 0} \frac{\dfrac{g(x)f(x+h) - f(x)g(x+h)}{g(x+h)g(x)}}{h}$$

$$= \lim_{h\to 0} \frac{g(x)f(x+h) - g(x)f(x) + g(x)f(x) - f(x)g(x+h)}{h.g(x+h)g(x)}$$

$$= \lim_{h\to 0} \frac{1}{g(x+h)g(x)} \left[\frac{g(x)\{f(x+h) - f(x)\} - f(x)\{g(x+h) - g(x)}{h}\right]$$

$$= \frac{1}{\{g(x)\}^2}\cdot\left[g(x)f'(x) - f(x)g'(x)\right] \text{ (by defn of } f'(x),\ g'(x)]$$

$$= \frac{g'(x)g(x) - g'(x)f(x)}{\{g(x)\}^2}$$

5.7 Differentiation of a function of a function (Chain Rule)

Let $y = f(u)$ where $u = g(x)$ such that $f(u)$ and $g(x)$ are continuous functions. Then y is also a continuous function of x.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

The above rule can be generalised

i.e. If $y = f(u)$ where $u = g(v)$ and $v = h(x)$

Then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$ and so on.

Cor. $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$ i.e. $\frac{dy}{dx} = \frac{1}{dx/dy}$

Provided non of $\frac{dy}{dx}$ or $\frac{dx}{dy}$ is zero

5.8 Differentiation of Inverse Circular Functions

(i) $\frac{d}{dx}(\sin^1 x) = \frac{1}{\sqrt{1-x^2}}, \ |x| \le 1$

Proof: Let $y = \text{Sin}^{-1}x$

$\therefore \quad x = \text{Siny}$

$\therefore \ \frac{dx}{dy} = \cos y$ [By result (v) of 5.5]

i.e $\frac{dx}{dy} = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$

$\therefore \ \frac{dx}{dy} = \frac{1}{dx/dy} = \frac{1}{\sqrt{1-x^2}}$ for $x \neq 1$, or -1

Thus $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \ \{-1 < x < 1\}$

(ii) $\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \ \{-1 < x < 1\}$

Proof: Same as proof of (i)

(iii) $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$

Proof: Let $y = \tan^{-1} x$ then $x = \tan y$

$\therefore \ \frac{dx}{dy} = \sec^2 y$ [by result (vii) of 5.5] i.e $\frac{dx}{dy} = 1 + \tan^2 y = 1 + x^2$

$\therefore \frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{1+x^2}$

Thus $\frac{d}{dy}(\tan^{-1} x) = \frac{1}{1+x^2}$

(iv) $\frac{d}{dx}(\cot^{-1} x) = -\frac{1}{1+x^2}$

Proof: Same as proof of (iii)

(v) $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$; $\{|x| > 1\}$

Proof: Let $y = \sec^{-1}x$. Then $x = \sec y$

$\therefore \frac{d}{dx} = \sec y \tan y$ [by result (ix) of 5.5]

i.e $\frac{dx}{dy} = \sec y \sqrt{\sec^2 y - 1}$

i.e $\frac{dx}{dy} = x\sqrt{x^2-1}$

$\therefore \frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{x\sqrt{x^2-1}}$ $x \neq 1$, or -1

Thus $\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$; $\{|x| > 1\}$

(vii) $\frac{d}{dy}(\operatorname{cosec}^{-1} x) = -\frac{1}{x\sqrt{x^2-1}}$; $\{|x| > 1\}$

Proof: Same as proof of (v)

5.8 Logarithmic Differentiation

Logarithmic Differentiation is used when we have functions raised to the power which is also a function or if we have the product of a number of functions. In such a situation it is convenient to take the logarithm of the expression and then differentiate it.

(i) Let $y = \{f(x)\}^{g(x)}$

Then $\log y = \log \{f(x)\}^{g(x)}$

i.e $\log y = g(x) \log \{f(x)\}$ $[\because \log a^m = m \log a]$

Differentiating both sides with respect to x

$$\frac{1}{y}\frac{dy}{dx} = g(x)\frac{1}{f(x)} f'(x) + g'(x) \log f(x)$$

[using result (iv) of 5.5; theorem 4 of 5.6 and chain rule]

$$\therefore \frac{dy}{dx} = y\left\{g(x)\frac{f'(x)}{f(x)} + g'(x) \log f(x)\right\}$$

$$\text{Thus } \frac{d}{dx}\{f(x)\}^{g(x)} = \{f(x)\}g(x)\left\{g(x)\frac{f'(x)}{f(x)} + g'(x) \log f(x)\right\}$$

(ii) Let $y = f_1(x) \times f_2(x) \times \ldots\ldots \times f_n(x)$

Then $\log y = \log \{f_1(x) \times f_2(x) \times \ldots\ldots \times f_n(x)\}$

i.e $\log y = \log f_1(x) + \log f_2(x) + \ldots\ldots + \log f_n(x)\}$

$[\because \log ab = \log a + \log b]$

Differentiating both sides w.r.f x

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{f_1(x)} \times f_1'(x) \times \frac{1}{f_2(x)} \times f_2'(x) + \ldots\ldots. + \frac{1}{f_n(x)} \times f_n'(x)$$

[Using result (iv) of 5.5, and Chain Rule]

$$\therefore \frac{dy}{dx} = y\left\{\frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} + \ldots\ldots\ldots + \frac{f_n'(x)}{f_n(x)}\right\}$$

$$\text{i.e } \frac{dy}{dx} = f_1'(x) . f_2(x)\ldots\ldots f_n(x) + f_2'(x) . f_1(x) . f_3(x)\ldots\ldots f_n(x)$$

$$+ f_3^1(x) . f_1(x) . f_2(x) . f_4(x)......f_n(x) +......$$

$$+ f_n^1(x) . f_1(x) . f_2(x)......f_{n-1}(x)$$

Thus $\frac{d}{dx}\{f_1(x) \times f_2(x) \times f_3(x) \times ... \times f_n(x)\}$

$$= f_1'(x) . f_2(x)......f_n(x) + f_2'(x) . f_1(x) . f_3(x).....f_n(x)$$

$$+......+ f_n'(x) . f_1(x) . f_2(x)......f_{n-1}(x)$$

5.9 Differentiation of Implicit Function

If f(x, y) = c be a function of x and y defined in such a way that y is not expressible directly in terms of x, then in such cases, the function is called an implicit function. In differentiating such functions, we differentiate both sides of the equation w.r.t x, regarding y as an unknown function of x having a derivative dy/dx.

i.e keeping in mind that

$$\frac{d}{dx} y^2 = 2y \frac{dy}{dx}; \frac{d}{dx} y^3 = 3y^2 \frac{dy}{dx} \text{ and so on.}$$

5.10 Differentiation of Parametric Equations

Sometimes x and y are expressed as a functions of a third variable known as a 'parameter'. In such cases to find $\frac{dy}{dx}$ it is not necessary to eliminate the parameter and express y in terms of x. We may proceed as follows:

Let $x = \phi(t)$ and $y = \psi(t)$

$$\text{Now } \frac{dy}{dx} = \frac{dy}{dt} . \frac{dt}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} \quad \left(\frac{dx}{dt} \neq 0\right)$$

Illustrative Examples

Example 1: Find from first principle, the derivative of $\sqrt{x}$ $(x > 0)$

Solution: Let $f(x) = \sqrt{x}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \text{ (by definition)}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \frac{\left[\sqrt{x+h} - \sqrt{x}\right]\left[\sqrt{x+h} + \sqrt{x}\right]}{h\left[\sqrt{x+h} + \sqrt{x}\right]}$$

$$= \lim_{h \to 0} \frac{\left(\sqrt{x+h}\right)^2 - \left(\sqrt{x}\right)^2}{h\left[\sqrt{x+h} + \sqrt{x}\right]}$$

$$= \lim_{h \to 0} \frac{x+h-x}{h\left[\sqrt{x+h} + \sqrt{x}\right]}$$

$$= \lim_{h \to 0} \frac{h}{h\left[\sqrt{x+h} + \sqrt{x}\right]} = \frac{1}{2\sqrt{x}}$$

Example 2: Find, from first principle, the differential coefficient of $\tan^{-1}x$.

Solution: Let $\tan^{-1}x = y$ and $\tan^{-1}(x+h) = y+k$

Then as $h \to 0$, $k \to 0$

Also $x = \tan y$, $x+h = \tan(y+k)$

$$\therefore \frac{d}{dx}(\tan^{-1}x) = \lim_{h \to 0} \frac{\tan^{-1}(x+h) - \tan^{-1}x}{h}$$

$$= \lim_{k \to 0} \frac{y+k-y}{\tan(y+k) - \tan y}$$

$$= \lim_{k \to 0} \frac{k}{\tan(y+k) - \tan y}$$

$$= \lim_{k \to 0} \frac{k}{\frac{\sin(y+k)}{\cos(y+k)} - \frac{\sin y}{\cos y}}$$

$$= \lim_{k \to 0} \frac{k \cos y \cos(y+k)}{\cos y \sin(y+k) - \sin y \cos(y+k)}$$

$$= \lim_{k\to 0} \frac{k \cos y \cos(y+k)}{\sin[(y+k)-y]}$$

$$= \lim_{k\to 0} \frac{k \cos y \cos(y+k)}{\sin k}$$

$$= \lim_{k\to 0} \frac{k}{\sin k} \cos(y+k) \cos y$$

$$= \cos^2 y = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$$

Example 3: A function is defined in the following way:

$$f(x) = |x|$$

i.e $f(x) = x$, when $x>0$

$= 0$, when $x=0$

$= -x$, when $x < 0$

Show that $f'(0)$ does not exists.

Solution: $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}$

Now $\lim_{h\to 0^+} \frac{f(h)}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$ and $\lim_{h\to 0^-} \frac{f(h)}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$

Since Right hand derivative is not equal to the left hand derivative $f'(0)$ does not exists.

Example 4: A function $f(x)$ is defined as follows

$f(x) = x \sin \frac{1}{x}$, $x\neq 0$

$f(0) = 0$

Show that $f'(0)$ does not exists.

Solution: $f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$

$$= \lim_{h\to 0} \frac{f(h)}{h}$$

$$= \lim_{h\to 0} \frac{h \sin \frac{1}{h}}{h}$$

$$= \lim_{h\to 0} \sin \frac{1}{h}, \text{ which does not exists}$$

f'(0) does not exists

Example 5: Find the differential coefficient of $\sin^2$ (log secx)

Solution: Let y = $[\text{Sin}(\log \sec x)]^2 = u^2$

Where u = sin (log secx) = sinv

Where v = log secx = log w

Where w = secx

$$\therefore \frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dv}.\frac{dv}{dw}.\frac{dw}{dx}$$

$$= 2u \,.\, \cos v \frac{1}{w}. \sec x \tan x$$

= 2 sin (log secx) cos (log secx) tanx

= sin (2log secx) tanx

Example 6: Differentiate $(\sec x)^{\tan x}$

Solution: Let y = $(\sec x)^{\tan x}$

logy = tany log secx

Differentiating both sides with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \tan x \frac{1}{\sec x} \sec x \tan x + \log \sec x \sec^2 x$$

$$= \tan^2 x + \sec^2 x \log \sec x$$

$$\therefore \frac{dy}{dx} = (\sec x)^{\tan x} [\tan^2 x + \sec^2 x \log \sec x]$$

Example 7: Find $\frac{dx}{dy}$ if $y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)}}$

Solution: Taking logarthim of both sides

$$\text{loy } y = \log \left\{\frac{(x-1)(x-2)}{(x-3)(x-4)}\right\}^{1/2}$$

i.e $\log y = \frac{1}{2}$ [log{(x–1) (x–2)} – log {(x–3) (x–4)}]

i.e $\log y = \frac{1}{2}$ [log{(x–1) + log (x–2) – log (x–3) – log (x–4)]

Differentiating both sides with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}\right]$$

$$= -\frac{2x^2-10x+11}{(x-1)(x-2)(x-3)(x-4)}$$

$$\therefore \frac{dy}{dx} = -\frac{2x^2-10x+11}{(x-1)^{1/2}(x-2)^{1/2}(x-3)^{3/2}(x-4)^{3/2}}$$

Example 8: Find $\frac{dx}{dy}$ if $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$

Solution: $\frac{dx}{d\theta} = a(1-\cos\theta)$, $\frac{dy}{d\theta} = a(-\sin\theta)$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{-a\sin\theta}{a(1-\cos\theta)} = \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2}$$

$$\therefore \frac{dy}{dx} = -\cot\theta/2$$

Example 9: If $\sin y = x\sin(a+y)$, then prove that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$

Solution: Given $\sin y = x\sin(a+y)$

$$\Rightarrow x = \frac{\sin y}{\sin(a+y)}$$

Differentiating both sides w.r.f to y

$$\frac{dx}{dy} = \frac{\sin(a+y)\cos y - \sin y\cos(a+y)}{\sin^2(a+y)}$$

$$\text{i.e } \frac{dx}{dy} = \frac{\sin(a+y-y)}{\sin^2(a+y)}$$

$$\therefore \frac{dx}{dy} = \frac{\sin a}{\sin^2(a+y)}$$

$$\text{Hence } \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

Example 10: using the definition of derivative, obtain the derivative of $\sin^2 x$ (NEHU 2013)

Solution: Here $f(x) = \sin^2 x$

$$\therefore f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin^2(x+h)-\sin^2 x}{h}$$

$$= \lim_{h\to 0} \frac{[\sin(x+h)-\sin x]\ [\sin(x+h)+\sin x]}{h}$$

$$= \lim_{h\to 0} \frac{2\cos\frac{x+h+x}{2}\sin\frac{x+h-x}{2}.2\sin\frac{x+h+x}{2}\cos\frac{x+h-x}{2}}{h}$$

$$= \lim_{h\to 0} \frac{2\cos\frac{2x+h}{2}\sin\frac{h}{2}\ .\ 2\sin\frac{2x+h}{2}\cos\frac{h}{2}}{h}$$

$$= 4\lim_{h\to 0}\cos\frac{2x+h}{2}\ \lim_{h\to 0}\frac{\sin h/2}{h}\ .\ \lim_{h\to 0}\sin\frac{2x+h}{2}\ \lim_{h\to 0}\cos\frac{h}{2}$$

$$= 4 \times \cos x \times \lim_{h\to 0}\frac{\sin h/2}{h/2 .2} .\sin x\times 1$$

$$= \frac{4}{2}\times \cos x\times 1\times \sin x = 2\sin x \cos x = \sin 2x$$

Example 11: Find $\frac{dy}{dx}$ if $x^y = y^x$ (NEHU 2006)

Solution: Given $x^y = y^x$

Taking logarithm of both sides we get

$y\log x = x\log y$ $[\because \log a^b = b\log a]$

Differentiating both sides w.r.t x we get

$$y.\frac{1}{x} + \log x\frac{dy}{dx} = \log x + x\frac{1}{y}\frac{dy}{dx}$$

$$\text{or } \frac{dy}{dx}\left[\log x - \frac{x}{y}\right] = \log y - \frac{y}{x}$$

$$\text{or } \frac{dy}{dx}\left[\frac{y\log x - x}{y}\right] = \left[\frac{x\log y - y}{x}\right]$$

$$\text{or } \frac{dy}{dx} = \frac{y}{x}\left[\frac{x\log y - y}{y\log x - x}\right]$$

Example 12: If $f(x) = \frac{1}{\phi(x)}, \phi(x) \neq 0$, then using the definition of derivatives,

prove that $f'(x) = \frac{-\phi'(x)}{[\phi(x)]^2}$ (NEHU 2010)

Solution: $f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0}\frac{\frac{1}{\phi(x+h)} - \frac{1}{\phi(x)}}{h}$$

$$= \lim_{h\to 0}\frac{\phi(x)-\phi(x+h)}{h\phi(x)\,\phi(x+h)}$$

$$= \lim_{h\to 0}\frac{\frac{-[\phi(x+h)-\phi(x)]}{h}}{\phi(x)\,\phi(x+h)}$$

$$= -\frac{\lim\limits_{h\to 0}\dfrac{\phi(x+h)-\phi(x)}{h}}{\lim\limits_{h\to 0}\phi(x)\,\phi(x+h)} = -\frac{\phi'(x)}{[\phi(x)]^2}$$

Exercise

Find the differential coefficients of the following:

1. (i) $\{\phi(x)\}^2$ (ii) $(x^2 + 5)^7$ (iii) $\sqrt{x^2+a^2}$ (iv) $(e^x)^2$

   (v) $\sqrt{\log x}$ (vi) $\tan^5 x$ (vii) $\sec^3 x$ (viii) $(\sin^{-1}x)^3$ (ix) $(\tan^{-1}x)^2$

2. (i) $e^{\phi(x)}$ (ii) e^{ax} (iii) $e^{(ax^2+bx+c)}$ (iv) e^{x^4} (v) $e^{\tan x}$ (vi) $e^{\sin^{-1}x}$

3. (i) $a^{\phi(x)}$ (ii) 7^{x^2+2x} (iii) $10^{\log_x^x}$

4. (i) $\log\,\phi(x)$ (ii) log sinx (iii) log cosx (iv) log (x+a)

   (v) $\log\sqrt{x}$ (vi) $\log(ax^2 + bx + c)$ (vii) log (logx)

   (ix) $10^{\log\sin x}$ (x) $\log\tan^{-1}x$ (xi) log (secx + tanx)

   (xii) $\log_x^a$ (xiii) $\log\left(\sqrt{x-a}+\sqrt{x-b}\right)$ (xiv) $\frac{1}{2}\log\frac{1+x}{1-x}$

5. (i) $\sin\,\phi(x)$ (ii) $\cos\,\phi(x)$ (iii) $\tan\,\phi(x)$ (iv) $\operatorname{cosec}\phi(x)$ (v) $\sec\,\phi(x)$ (vi) $\cot\,\phi(x)$ (vii) cos (ax+b) (viii) $\operatorname{cosec}^3 x$ (ix) sin2x cosx

   (x) cos2x cos3x (xi) $\sin x^0$ (xii) $e^{ax}\sin bx$ (xiii) $e^{ax}\cos bx$

6. (i) $\sin^{-1}\phi(x)$ (ii) $\tan^{-1}\phi(x)$ (iii) $\sin^{-1}x^2$ (iv) $\tan^{-1}\sqrt{x}$

   (v) $\sin^{-1}\frac{x}{a}$ (vi) $\sec^{-1}x^3$ (vii) $\cot^{-1}(e^x)$ (viii) $\sec^{-1}(\tan x)$

   (ix) $\tan^{-1}(\sec x)$ (x) $\sin^{-1}(3x - 4x^3)$ (xi) $\sec(\tan^{-1}x)$

   (xii) $\tan(\sin^{-1}x)$ (xiii) $\cos^{-1}\frac{1-x^2}{1+x}$ (xiv) $\sin^{-1}\frac{2x}{1+x^2}$

   (xv) $\tan^{-1}\frac{2x}{1-x^2}$ (xvi) $\tan^{-1}\frac{1}{\sqrt{x^2-1}}$ (xvii) $\tan^{-1}\frac{3x-x^2}{1-3x^2}$

7. (i) $\cos\left\{\sqrt{1+x^2}\right\}$ (ii) $e^{\sqrt{\cot x}}$ (iii) $e^{\operatorname{cosec}^2\sqrt{x}}$ (iv) $e^{(\sin^{-1}x)^2}$

(v) $\sqrt{\log\sin x}$ (vi) $\cos\{2\sin^{-1}\cos x\}$ (vii) $\sin^2(\log x^2)$

8. (i) x^x (ii) $x^{\log x}$ (iii) a^{a^x} (iv) e^{e^x} (v) $(\sin x)^{\tan x}$ (vi) $x^{\cos^{-1}x}$

(vii) $(\sin)^{\log x}$ (viii) $(\sin x)^{\cos x} + (\cos x)^{\sin x}$

(ix) $(\tan x)^{\cot x} + (\cot x)^{\tan x}$ (x) x^{x^x}

9. (i) $(1-x)(1-2x)(1-3x)(1-4x)$ (ii) $\sqrt[3]{x(x+1)(x+2)}$

(iii) $\sqrt{\dfrac{1+x}{1-x}}$ (iv) $\log\left\{e^x\left(\dfrac{x-1}{x+1}\right)^{3/2}\right\}$

10. Find $\dfrac{dy}{dx}$ in the following cases:

(i) $3x^4 - x^2y + 2y^3 = 0$ (i) $x^4 + x^2y^2 + y^4 = 0$

(iii) $x^3 + y^3 = 3axy$ (iv) $x^{2/3} + y^{2/3} = a^{2/3}$ (v) $x = y\log xy$

(vi) $y = x^y$ (vii) $x^y = y^x$ (viii) $x^y.y^x = 1$ (ix) $e^{xy} - 4xy = 2$

(x) $\log(xy) = x^2 + y^2$

11. Find $\dfrac{dy}{dx}$ when :

(i) $x = a\cos\phi,\ y = b\sin\phi$

(ii) $x = a\cos^3\theta,\ y = b\sin^3\theta$

(iii) $x = at^2,\ y = 2at$

(iv) $x = a(\cos t + \log\tan\frac{1}{2}t),\ y = a\sin t$

(v) $x = a(\cos t + t\sin t),\ y = a(\sin t - t\cos t)$

(vi) $x = 2a\sin^2 t\cos 2t,\ y = 2a\sin^2 t\sin 2t$

(vii) $x = \dfrac{3at}{1+t^3},\ y = \dfrac{3at}{1+t^3}$

(viii) $\tan y = \dfrac{2t}{1-t^2},\ \sin x = \dfrac{2t}{1+t^2}$

12. If $y = e^{\sin^{-1}x}$ and $z = e^{-\cos^{-1}x}$, then show that $\frac{dy}{dz}$ is independent of x.

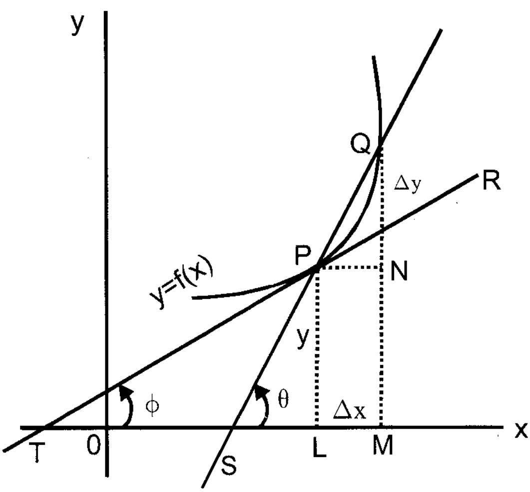
13. Differentiate the left side functions with respect to the right side ones

(i) x^5 w.r.t x^2 (ii) secx w.r.t tanx (iii) $\log_{10}^{x}$ w.r.t x^3

(iv) $\tan^{-1}x$ w.r.t x^2 (v) $\cos^{-1}\frac{1-x^2}{1+x^2}$ w.r.t $\tan^{-1}\frac{2x}{1-x^2}$

(vi) $x^{\sin^{-1}x}$ w.r.t $\sin^{-1}x$

5.11. Geometrical Significance of derivative and its sign.



Let P(x, y) be any point on the curve y=f(x) and Q(x + Δx, y + Δy) be a negighbouring point of P. Let TPR be a tangent at P making an angle ϕ with the positive direction of x-axis. Let PQ be a chord joining P and Q and making an angle θ with the positive direction of X–axis

Equation of the line PQ is

$$Y-y = \frac{Y+\Delta y - y}{x+\Delta x - x}(X-x)$$

$$\text{i.e } Y-y = \frac{\Delta y}{\Delta x}(X-x)$$

But $\frac{\Delta y}{\Delta x}$ = $\tan\theta$, the slope of the line PQ

Now, if Q approaches P along the curve indefinitely, Closely $\Delta x \to 0$ and the chord PQ tends to a definite limiting position TPR which is the tangent to the curve at P; an as PQ tends to TPR, $\theta \to \phi$

Also as $\Delta x \to 0$, $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$ from definition

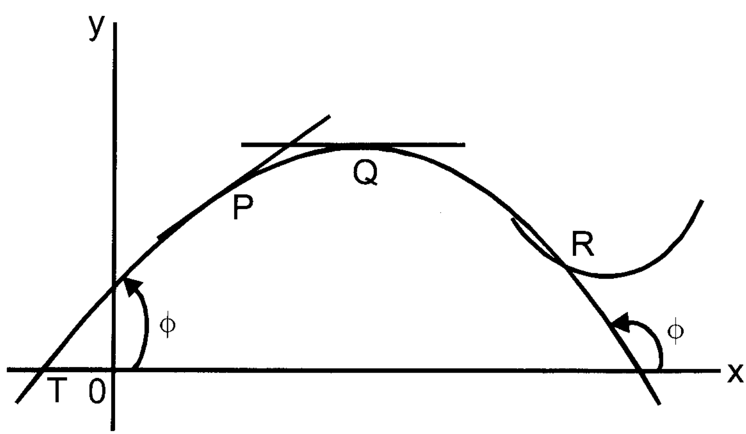
Hence $\tan\phi = \frac{dy}{dx}$. Thus the derivative $\frac{dy}{dx}$ for any value of x, if exists represents the trigonometrical tangent of the angle of inclination (kown as slope or gradient) of the tangent line at the corresponding point P on the curve y=f(x)

5.12 Signs of derivative

(i) If $\frac{dy}{dx}$ (= $\tan\phi$) > 0, ϕ is acute and at that point y increases with x.

(ii) If $\frac{dy}{dx}$ (= $\tan\phi$) < 0, ϕ is obtuse and at that point y decreases when x incrreases or vice versa

(iii) If $\frac{dy}{dx}$ (= $\tan\phi$) < 0, the tangent line is parallel to x-axix



Illustrative Example

Example 1: Find the points on the curve $y = 2x^3 - 3x + 5$ where the tangent is parallel to x-axis. (NEHU, 2003)

Solution: Since the tangent is parallel to x-axis

$$\frac{dy}{dx} = 0$$

i.e $\frac{d}{dx}(2x^3 - 3x + 5) = 0$

i.e $6x^2 - 3 = 0$

$\Rightarrow 3(2x^2 - 1) = 0 \Rightarrow 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

Putting $x = \pm\frac{1}{\sqrt{2}}$ in $y = 2x^3 - 3x + 5$ we get

$y = 2\left(\frac{1}{\sqrt{2}}\right)^3 - 3\left(\frac{1}{\sqrt{2}}\right) + 5$ or $y = 2\left(-\frac{1}{\sqrt{2}}\right)^3 - 3\left(-\frac{1}{\sqrt{2}}\right) + 5$

i.e $y = \frac{1}{\sqrt{2}} - \frac{3}{\sqrt{2}} + 5$ or $y = -\frac{1}{\sqrt{2}} + \frac{3}{\sqrt{2}} + 5$

i.e $y = \frac{5\sqrt{2} \pm 2}{\sqrt{2}}$

Hence, the tangent is parallel to x axis at the point $\left(\pm\frac{1}{\sqrt{2}}, \frac{5\sqrt{2} \pm 2}{\sqrt{2}}\right)$

Example 2: Determine the slope of the tangent to the curve $y = \frac{x(x^2 - 1)}{x^2 + 1}$ at the origin. (NEHU 2004)

Solution: The slope of the tangent to the curve is given by $\frac{dy}{dx}$

Now $y = \frac{x(x^2 - 1)}{x^2 + 1}$

$$\therefore \frac{dy}{dx} = \frac{(x^2+1)(3x^2-1)-.x(x^2-1).2x}{(x^2+1)^2}$$

$$\text{i.e } \frac{dy}{dx} = \frac{3x^4-x^2+3x^2-1-2x^4+2x^2}{(x^2+1)^2}$$

$$\text{i.e } \frac{dy}{dx} = \frac{x^4+4x^2-1}{(x^2+1)^2}$$

At the origin (0, 0) $\left[\frac{dy}{dx}\right]_{(x,y)=(0,0)} = -1$

Example 3: Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangents are parallel to X-axis. (NEHU, 2006)

Solution: $\frac{dy}{dx} = \frac{d}{dx}(2x^3-3x^2-12x+20) = 6x^2-6x-12$

Since the tangents are parallel to X axis

$$\frac{dy}{dx} = 0 \Rightarrow 6x^2 - 6x - 12 = 0$$

$$\Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow (x-2)(x+1) = 0$$

$$\Rightarrow x = -1, 2$$

Putting the values of $x = -1, 2$ in $y = 2x^3-3x^2-12x+20$

We get $y = 2(-1)^3-3(-1)^2-12(-1)+20$

or $y = 2(2)^3 - 3(2)^2 - 12(2)+20$

i.e $y = 27, -15$

The tangents are parallel to X axis at the point (–1, 27) and (2, –15)

6

Applications of Derivative

6.1 Derivative as a Rate Measurer

Let $y = f(x)$

Then $\frac{dy}{dx}$ denotes the rate of change of y w.r.t x and its value at x=a is denoted by $\left[\frac{dy}{dx}\right]_{x=a}$

Illustrative Examples

Example 1: Find the rate of change of the area of a circle with respect to its radius r when r = 6cm

Solution: Let A be the area of the circle of radius r. Then

$$A = \pi r^2 \Rightarrow \frac{dA}{dr} = \frac{d}{dr}\left(\pi r^2\right) = 2\pi r$$

$$\therefore \left[\frac{dA}{dr}\right]_{r=6cm} = 2\pi \times 6 = 12\pi \text{ cm}^2/\text{cm}$$

Hence the area is changing at the rateof 12π cm²/cm

Example 2: If the area of a circle increases at a uniform rate, show that the rate of increase of the perimeter varies inversely as the radius.

(NEHU 2008, 2013)

Solution: Let A be the area at any time t, P the perimeter and r the radius of the circle

Then $A = \pi r^2$; $P = 2\pi r$

$\therefore P^2 = 4\pi^2 r^2 = 4\pi A$

Differentiating both sides w.r.t t, we have

$$2P \frac{dp}{dt} = 4\pi \frac{dA}{dt}$$

$$\text{i.e } \frac{dp}{dt} = \frac{d\pi}{P}\frac{dA}{dt} = \frac{1}{r}\frac{dA}{dt}$$

$$\text{i.e } \frac{dp}{dt} \infty \frac{1}{r} \quad [\because \frac{dA}{dt} \text{ is constant}]$$

Hence the perimeter varies inversely as the ractius.

Example 3: A balloon which always remains spherical has a variable radius. Find the rate at which its volume is increasing with the radius, when the radius, is 10 cm. (NEHU 2005)

Solution: Let V be the volume of the balloon at any instant of time t, and r its radius

Then $V = \frac{4}{3}\pi r^3$

Differentiating both sides w.r.t r we have

$$\frac{dv}{dr} = \frac{4}{3}\pi 3r^2 \frac{dr}{dt} = 4\pi r^2$$

$$\text{when } r = 10 \text{ cm, } \frac{dv}{dr} = 4\pi \times 0^2 = 400\pi$$

Hence the volume is increasing at the rate of $400\pi cm^2$ per sec.

Example 4: The area of an expanding rectangle is increasing at the rate of 48 cm^2/sec. The length of the retangle is always equal to the square of the breadth. At what rate the length is increasing at the instant when the breadth is 4.5 cm. (NEHU 2008)

Solution: Let A be the area of the rectangle at any instant of time 't' and *l* the length and b the breadth of the rectangle.

Then $A = l \times b$ ....(1)

and $l = b^2$ ....(2)

$\therefore A = l\sqrt{l}$ ....(3)

Given $\frac{dA}{dt} = 48$

Differentiating (3) w.r.t t we have

$$\frac{dA}{dt} = \frac{3}{2}\sqrt{l}\ \frac{dl}{dt}$$

i.e $\quad 48 = \frac{3}{2}\sqrt{l}\ \frac{dl}{dt}$

i.e $\quad 32 = \sqrt{l}\ \frac{dl}{dt}$

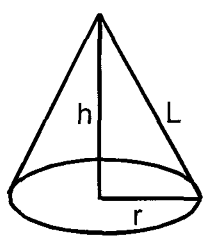
Now when b = 4.5, $l = (4.5)^2 = 20.25$

$$\therefore \frac{dl}{dt} = \frac{32}{\sqrt{l}} = \frac{32}{\sqrt{20.25}} = \frac{32}{4.5} = \frac{320}{45} = \frac{64}{9}$$

Hence the length in increasing at the rate of $\frac{64}{9}$ cm/sec

Example 5: The volume of a right circular cone remains constant. If the radius of the base is increasing at the rate of 3 cm per second, then how fast is the altitude changing when the altitude is 8cm and radius 6 cm. (NEHU, 2014)

Solution: Let V be the volume, h the altitude and r the radius of the base of the cone at any instant of time t.



Now $V = \frac{1}{3}\pi r^2 h$ = constant = k ....(i)

Also $\frac{dr}{dt} = 3$ ...... (ii)

from (i) $h = \frac{3k}{\pi r^2}$

Differentiating both sides w.r.t t we have

$$\frac{dh}{dt} = \frac{3k}{\pi}\frac{d}{dt}\left(\frac{1}{r^2}\right)$$

i.e $\dfrac{dh}{dt} = \dfrac{3k}{\pi}\left(-\dfrac{2}{r^3}\right)\dfrac{dr}{dt}$

i.e $\dfrac{dh}{dt} = \dfrac{3k}{\pi}\left(-\dfrac{2}{r^3}\right)\times 3$ by (ii)

When r = 6, $\dfrac{dh}{dt} = -\dfrac{18k}{\pi}\times\dfrac{1}{216} = -\dfrac{k}{12\pi}$

Hence altitude decreases at the rate of $\dfrac{k}{12\pi}$ cm/sec

Example 6: A store is dropped into a quiet lake and the waves move in circles. If the radius of a circular wave increases at the rate of 4 cm/sec, find the rate of increase in its area at the instant when its radius is 10 cm.

Solution: Let r be the radius and A the area if the circle at any instant t

Now given $\dfrac{dr}{dt} = 4$ ....(i)

Then $A = \pi r^2 \Rightarrow \dfrac{dA}{dt} = 2\pi r\dfrac{dr}{dt}$

$\Rightarrow \dfrac{dA}{dt} = 2\pi r.4$ by (i)

i.e $\dfrac{dA}{dt} = 8\pi r$

When r = 10, $\dfrac{dA}{dt} = (8\pi)(10) = 80\pi$

Hence the area is increasing at the rate of 80π cm²/sec

Example 7: The volume of a spherical ballon is increasing at the rate of 20 cm³/sec. Find the rate of change of its surface area at the instant when its radius is 8 cm.

Solution: Let V be the volume. S the surface and r the radius of the ballon at any instant of time t.

Then $V = \dfrac{4}{3}\pi r^3$, $S = 4\pi r^2$ ....(1)

Given that $\frac{dV}{dt} = 20$ ....(2)

i.e $\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 20$

i.e $\frac{4}{3}\pi 3r^2 \frac{dr}{dt} = 20$

i.e $\frac{dr}{dt} = \frac{5}{\pi r^2}$ .... (3)

Now $S = 4\pi r^2 \Rightarrow \frac{ds}{dt} = 4\pi\ 2r\ \frac{dr}{dt}$

$\Rightarrow \frac{ds}{dt} = 8\pi r.\frac{5}{\pi r^2} = \frac{40}{r}$ by (3)

When $r = 8$, $\frac{ds}{dt} = \frac{40}{8} = 5$

Hence the surface area is increasing at the rate of 5 cm^2/sec

Example 8: A 5m ladder is leaning against a wall. The botton of the ladder is pulled along the ground away from the wall at the rate of 2m/sec. How fast is its height on the wall decreasing. When the foot of the ladder is 4 cm away from the wall?

Solution: Let AB be the position of the ladder at any instant of time 't' and OB is the wall such that OA = x and OB = y

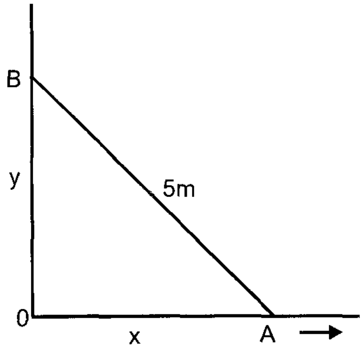
Given that AB = 5, $\frac{dx}{dt} = 2$ ....(1)

Now $x^2 + y^2 = 5^2 \Rightarrow y^2 = 25 - x^2$

$\therefore \frac{d}{dt}y^2 = \frac{d}{dt}\left(25 - x^2\right)$

i.e $2y\frac{dy}{dt} = -2x\frac{dx}{dt}$

i.e $\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = \frac{-2x}{y}$ by (1)



Now when x = 4, y = $\sqrt{25-4^2}$ =3

$$\therefore \frac{dy}{dt} = -\frac{2.4}{3} = -\frac{8}{3}$$

Hence the height of the ladder on the wall is dereasing at the rate of $\frac{8}{3}$ cm/sec

Example 9: A particle moves along the curve $6y = x^3+2$. Find the point on the curve at which the y cordinate is changing 8 times as fast as the x-coordinate

Solution: Let (x, y) be the position of the particle on the curve at any instant 't'.

Given $\frac{dy}{dt} = 8\frac{dx}{dt}$ ....(1)

Given curve is $6y = x^3 + 2$ ....(2)

Differentiating both sides w.r.t t we have

$$6\frac{dy}{dt} = 3x^2\frac{dx}{dt}$$

i.e $6\left(8\frac{dx}{dt}\right) = 3x^2\frac{dx}{dt}$ by (1)

i.e $48 = 3x^2 \Rightarrow x^2 = 16$ i.e $x = \pm 4$

Putting $x = \pm 4$ in (2) we get $y = 11, -\frac{31}{3}$

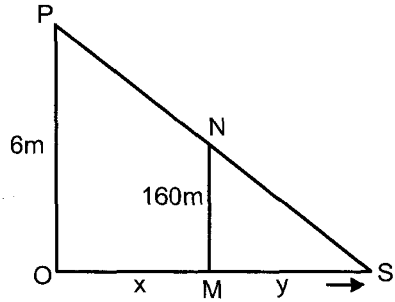
Hence the required points are (4, 11) and $(-4, -\frac{31}{3})$

Example 10: A man 160cm tall, walks away from a source of light situated at the top of a pole 6m high, at the rate 1.1 m/sec. How fast is the length of his shadow increasing when he is 1m away from the pole.

Solution: Let MN be the position of the man and MS be the length of his shadow at any instant 't'. Let OM = x, MS = y

Given that $\frac{dx}{dt} = 1.1$ ....(1)

Now from similar Δ's POS, NMS we have



$$\frac{PO}{MN} = \frac{OS}{MS} = \frac{6}{1.6}$$

i.e $\frac{x+y}{y} = \frac{15}{4}$

i.e $4x + y = 15y \Rightarrow x = \frac{11}{4}y$

$\therefore \frac{dx}{dt} = \frac{11}{4}\frac{dy}{dt}$

i.e $1.1 = \frac{11}{4}\frac{dy}{dt}$

$\Rightarrow \frac{dy}{dt} = \frac{4 \times 1.1}{11} = 0.4$

Hence the shadow increases at the rate of 0.4 m/sec

6.2 Errors and Approximations

Let $y = f(x)$ then $\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x)$

$\therefore \frac{f(x+\Delta x)-f(x)}{\Delta x} = f'(x) + \in$, where $\in \to 0$ as $\Delta x \to 0$

$\Rightarrow f(x + \Delta x) - f(x) = \Delta x\ f'(x) + \in . \Delta x$

$\Rightarrow f(x + \Delta x) - f(x) = \Delta x\ f'(x)$ (approximately)

$\therefore f(x + \Delta x) = f(x) + \Delta x\ f'(x)$ (approximately)

or $\Delta y = \Delta x\ f'(x)$ (approximately) $[\because f(x + \Delta x) - f(x) = \Delta y]$

Thus if Δx is an error in x, then the corresponding error in y is Δy. These small values Δx, Δy are called differentials.

Absolute Error: Δx is called an absolute error in x

Relative Error: $\frac{\Delta x}{x}$ is called the relative error

Percentage Error: $\frac{\Delta x}{x} \times 100$ is called the percentage error.

Illustrative Examples

Example 1: Find dx, Δy, Δy–dy, given that $y = \frac{x^2}{2} + 3x$, x = 2 and Δx = 0.5

Solution: dx = Δx = 0.5

$$\Delta y = f(x + \Delta x) - f(x)$$

$$= (y + \Delta y) - y$$

$$= \left\{\frac{(2.5)^2}{2} + 3\times 2.5\right\} - \left\{\frac{2^2}{2} + 3\times 2\right\}$$

$$= 2.625$$

$$dy = \frac{dy}{dx}\,dx = (x + 3)\,dx = (2+3)\times 0.5 = 2.5$$

$\therefore\ \Delta y - dy = 2.625 - 2.5 = 0.125$

Example 2: Using differentials, find the approximate value of $(127)^{1/3}$ upto three places of decimals

Solution: Let $(127)^{1/3} = (125+2)^{1/3} = (x+\Delta x)^{1/3}$ ....(1)

Where x = 125, Δx = 2

Consider $f(x) = x^{1/3}$, then

$f(x + \Delta x) = (x+\Delta x)^{1/3} = (127)^{1/3}$ by (1)

Now $f'(x) = \frac{1}{3x^{2/3}}$

Hence using $f(x + \Delta x) = \Delta x f'(x) + f(x)$ we get

$$(127)^{1/3} = 2\times\frac{1}{3(125)^{2/3}} + (125)^{1/3}$$

$$= 2\times\frac{1}{3(5^3)^{2/3}} + (5^3)^{1/3}$$

$$= \frac{2}{75} + 5$$

$$= \frac{377}{75} = 5.026 \text{ (approximately)}$$

Example 3: Given that $\log_{10}^{e} = 0.4343$, find the approximate value of $\log_{10}^{10.1}$

(NEHU 2002)

Solution: Let $\log_{10}^{10.1} = \log_{10}(10 + 0.1)$

$= \log_{10}(x + \Delta x)$ ....(1)

Where $x = 10$ and $\Delta x = 0.1$

Consider $f(x) = \log_{10}x$, then

$f(x + \Delta x) = \log_{10}(x + \Delta x) = \log_{10}^{10.1}$ by (1)

Now $f'(x) = \frac{1}{x} \times \log_{10}^{e}$ $\left[\because \log_{10}^{x} = \log_{e}^{x} \times \log_{10}^{2}\right]$

Hence using $f(x + \Delta x) = f(x) + \Delta x\, f'(x)$ (approx) we get

$$\log_{10}^{10.1} = \log_{10}^{10} + (0.1) \times \frac{1}{10} \times \log_{10}^{e}$$

$= 1 + 0.01 \times 0.4343$

$= 1.004343$ (approximately)

Hence $\log_{10}^{10.1} = 1.004343$

Example 4: If the radius of a circle increases from 5 cm to 5.1 cm, find the increase in area.

Solution: Area of the circle of radius r is given by

$A = \pi r^2$

Now $A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r$

When $r = 5$, $\frac{dA}{dr} = 10\pi \text{ cm}^2$

Also $\Delta r = (5.1 - 5)$ cm $= 0.1$ cm

$\therefore \Delta A = \frac{dA}{dr} \cdot \Delta r = (10\pi \times 0.1)\text{ cm}^2 = \pi$ cm

Hence increase in area is π cm^2

Example 5. The time T of oscellation of a simple Pendulum of length l is given by T = $2\pi\sqrt{\frac{l}{g}}$. Find the percentage error in T, corresponding to an error of 2% in the value of l.

Solution: T = $2\pi\sqrt{\frac{l}{g}}$

$$\Rightarrow \log T = \log 2 + \log\pi + \frac{1}{2}\log l - \frac{1}{2}\log g$$

$$\Rightarrow \frac{1}{T}\frac{dT}{dl} = \frac{1}{2l}$$

$$\Rightarrow \frac{1}{T}\frac{dT}{dl}\Delta l = \frac{1}{2l}\,\Delta l$$

$$\Rightarrow \frac{1}{T}\,\Delta T = \frac{1}{2l}\,\Delta l \quad \left[\because \Delta T = \frac{dT}{dl}\,\Delta l\right]$$

$$\Rightarrow \frac{\Delta T}{T}\times 100 = \frac{1}{2}\left(\frac{\Delta l}{l}\times 100\right) = \frac{1}{2}\times 2 = 1 \quad \left[\because \frac{\Delta l}{l} = \frac{2}{100}\right]$$

$\therefore$ error in T is 1% corresponding to an error of 2% in l.

Example 6. Find the approximate value of $\log_{10}^{404}$ by the use of differential, given that $\log_{10}^{4} = 0.6021$, $\log_{10}^{e} = 0.4343$ (NEHU 2005)

Solution : Same as example 3.

Example 7. Use differential to Compute the approximate value of $\log_{e}^{10.1}$. Given that $\log_{e}^{10} = 2.303$ (NEHU 2001)

Solution : Same as example 3.

Example 8. Find the approximate value of Sin62^0 by the method of differentials. Given that Sin 60^0 = 0.86603 and 1^0 = 0.0175 radiaus. (NEHU 2018)

Solution : Let Sin 62^0 = Sin (60^0 + 2^0) = Sin(x+Δx) ..... (1)

where x = 60^0 and Δx = 2^0

Consider f(x) = Sinx = $\text{Sin}\frac{\Delta x}{180}$

f(x + Δx) = Sin (x + Δx) = Sin62^0 by (1)

Now $f'(x) = \frac{\pi}{180} \text{Cos} \frac{\pi x}{180}$

Hence using $f(x + \Delta x) = f(x) + f'(x)$ (approx) we get

$\text{Sin}62^0 = \text{Sin}60^0 + 2^0 \times \text{Cos}60^0$

$= 0.86603 + 2 \times 0.0175 \times \frac{1}{2}$

$= 0.883853$

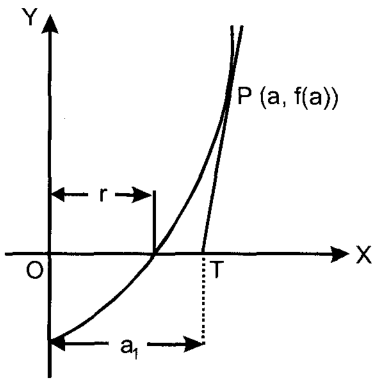
Hence $\sin 62^0 = 0.883853$ (approx)

6.3. Determination of Roots

Newton's Method of approximating a root

Newton's method of approximations depends on having at our disposal some device by which we can locate a root by trial. The method of trial depends on the following property of Continuty :

If a continuous function f(x) changes its signs in the interval [a, b] and if its derivative does not change sign, then the equation f(x) = 0 has one and only one root between a and b.



If the graph of y = f(x) cuts x-axis at different points, the intercepts on x-axis are the roots of the equation f(x) = 0

Consider the curve y = f(x) represented by the figure.

Clearly x = r, the intercept of the curve on x-axis, is a root of f(x) = 0. The exact determination of the value of r may not always be possible, specially when r is an irrational number.

Suppose by trial, we obtain an approximate value of the root r of f(x) as indicated in the figure.

If a be sufficiently close to r, the tangent line PT drawn at the point (a, f(a)) will have an x-intercept a_1 (say). this velue a_1 is in general more approximate to r than a. The equation of the tangent line PT whose slope is f'(a) is

$$y - f(a) = f'(a)(x-a)$$

Its x intercept a_1 which is obtained by putting y = 0 and solving for x is

$$a_1 = a - \frac{f(a)}{f'(a)} \qquad .....(A)]$$

Having found a_1, by (A) we may substitute a_1 for a in the RHS of (A) and obtain

$$a_2 = a_1 - \frac{f(a_1)}{f'(a_1)}$$

Then a_2 is a better approximation to the exact root r. Newton's method of approximation of the root of f(x) = 0 consists in using the formula (A) repeatedly.

Illustrative Examples

Example 1: Find the value of the real roots of the equation $f(x) \equiv x^3 - 5x - 5 = 0$ Correct ot 4 decimal places by using Newton's formula for approximation.

Solution: Since 3 can be taken as an approximate root

We substitute a = 3 in (A)

$$\therefore a_1 = 3 - \frac{f(3)}{f^1(3)} = 3 - \frac{3^3 - 5.3 - 5}{3.3^2 - 5} = 2.7 \text{ approx}$$

We now put a = 2.7 in (A) for better approximation

$$\therefore a_2 = 2.7 - \frac{f(2.7)}{f'(2.7)} = 2.7 - \frac{1.183}{16.87} = 2.63 \text{ approx.}$$

$$\text{Similarly } a_3 = 2.63 - \frac{f(2.63)}{f'(2.63)} = 2.63 - 0.00263$$

$$= 2.62737$$

$$= 2.6274 \text{ approx}$$

Example 2. Approximate the roots of $2\text{Cos}x - x^2 = 0$

Solution : Draw the graph of $y = 2\cos x$ and $y = x^2$

The two curves will be found to intersect in two points whose absussae are approximately 1 and – 1 put a = 1 in (A) then

$$a_1 = 1 - \frac{2\cos 1 - 1}{-2\sin 1 - 2} = 1 + \frac{2(0.5403) - 1}{2(0.8415) + 2} = 1.02$$

Now put a = 1.02 in (A) we get

$$a_2 = 1.02 - \frac{2\cos(1.02) - (1.02)^2}{-2\sin(1.02) - 2(1.02)}$$

$$= 1.02 + \frac{0.0064}{3.7442} = 1.0217$$

thus the roots are 1.0217 and – 1.0217 upto 4 decimal places

[Note that in an even function if r is one root then – r is the other]

Exercises

1. A point moves on the parabola $3y = x^2$ in such a way that when x = 3. the abscissa is increasing at the rate of 3 cm per second. At what rate is the ordinate increasing at that point?
2. If the rate of change of y with respect ot x is 5 and x is changing at 3 units per second, how fast is y changing?
3. The radius of a sphere increases at the rate of 7 cm per second. Find the rate at which the volume of the sphere increases.
4. The side of a square is increasing at hte rate of 0.2 cm per second. Find the rate of increase of the perimeter of the square.
5. A circular plate of metal expands by heat so that its radius increases at the rate of 0.25 cm per second. Find the rate at which the surface area is increasing when the radius is 7 cm.
6. The side of a square sheet of metal is increasing at 3 cm per minute. At what rate is the area increasing when the side is 10 cm long?
7. The radius of a circular soap bubble is increasing at the rate of 0.2 cm per second. Find the rate of increase of its surface area when the radius is 7 cm.
8. The radius of an air bubble is increasing at the rate of 0.5 cm per second. At what rate is the volume of the bubble increasing when the radius is 1 cm?
9. The volume of a spherical balloon is increasing at the rate of 25. cubic

centimetres per second. Find the rate of change of its surface at the instant when its radius is 5 cm.

10. A balloon which always remains spherical is being inflated by pumping in 900 cubic centimetres of gas per second. Find the rate at which the radius of the balloon is increasing, when the radius is 15 cm.

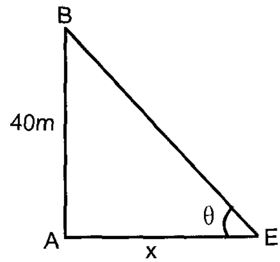
11. Δ stone is dropped into a quiet lake and waves move in circles at a speed of 3.5 cm per second. At the instant when the radius of the circular wave is 7.5 cm, how fast is the enclosed area increasing? (Take $\pi = 22/7$)

12. A 2 m tall man walks at a uniform speed of 5 km per hour away from a 6 m high lamp post. Find the rate at which the length of his shadow increases.

13. An inverted cone has a depth of 40 cm and a base of radius 5 cm. Water is pouned into it at the rate of 1.5 cubic centimetres per minute. Find the rate at which the level of the water in the cone, is rising when the depth is 4 cm.

14. A 13 m long ladder is leaning against a wall. the bottom of the ladder is pulled along the ground away from the wall at the rate of 2m per second. How fast is its height on the wall decreasing when the foot of the ladder is 5m away from the wall?

15. A man is moving away from a 40m high tower at a speed of 2m per second. Find the rate at which the angle of elevation of the top of the tower is changing when he is at a distance of 30m from the foot of the tower. Assume that the eye level of the man is 1.6m from the ground.



16. An edge of a variable cube is increasing at the rate of 5cm per second. How fast is the volume of the cube increasing when the edge is 10cm long?

17. Using differentials, find the approximate values of the following:

(i) $\sqrt{37}$ (ii) $\sqrt[3]{29}$ (iii) $\sqrt[3]{27}$ (iv) $\sqrt{0.24}$

(v) $\sqrt{0.48}$ (vi) $\sqrt[4]{15}$ (vii) $\frac{1}{(2.002)^2}$

(viii) $\log_e^{10.02}$ given that $\log_e^{10} = 2.3026$

(ix) $\log_{10}^{4.04}$, it being given that $\log_{10}^{4} = 0.6021$ and $\log_{10}^{e} = 0.4343$.

(x) $\cos 61^0$, it being given that $\sin 60^0 = 0.86603$ and $1^0 = 0.01745$ radian.

18. If the length of a simple pendulum is decreased by 2%, find the percentage decrease in its period T, where $T = 2\pi\sqrt{\frac{l}{g}}$

19. Using Newton's formula for approximation, calculated the root of the following:

    (i) $\cos x - x = 0$ (ii) $\cos x + x = 0$ (iii) $3\sin x - x = 0$

    (iv) $e^x + x - 3 = 0$ (v) $x^3 + 2x - 8 = 0$ (vi) $x^3 - 40 = 0$

20. One root of the equation $x^4 + 10x - 100 = 0$ is approximately equal to 3. Find its roots correct to two decimal places.

7

Successive Differentiation

7.1 The derivative of the function f(x) denoted by $\frac{dy}{dx}$, y_1, y', $f'(x)$ or D f(x) is in general a function of x. This derivative may again be a derivable function of x, which is called a second derivative (or second differential coefficient) of f(x) denoted by $\frac{d^2y}{dx^2}$, y_2, y'', $f''(x)$ or D^2 f(x)

Similarly the second derivative, may again be differentiable to give the third derivative (or third differential coefficient) and so on.

The nth derivatives are denoted by the symbols y_n, $y^{(n)}$, $f^{(n)}(x)$, D^n f(x) etc.

In general although successive derivatives can be found one by one as far as necessary, it is not always possibe to obtain an expression of the nth derivative. In some cases, however, it is possible by careful inspection of the first few derivatives to "infer a law of formation" which will permit an explicit formula to be written for the nth derivative. Strictly speaking, the nth derivative are to be established generally by the method of induction.

7.2 The nth derviatives of some special function:

(i) $y = x^n$, where n is a positive integer

By actual differentiation

$y_1 = nx^{n-1}$, $y_2 = n(n-1)x^{n-2}$. $y_3 = n(n-1)\ (n-2)\ x^{n-3}$

Proceeding in a similar manner we have

$y_r = n(n-1)(n-2)(n-3)......\{[n-(r-1)]\}x^{n-r}\ (r<n)$

$y_n = n(n-1)(n-2)(n-3)......3.2.1 = n!$

i.e $D^n(x^n) = n!$

Cor: Since $y_n = n!$ which is a constant $y_{n+1}, y_{n+2},....$are all zeroes in this case

(ii) $y = (ax+b)^m$ where m is any number

By actual differentiation

$y_1 = ma(ax+b)^{m-1},\ y_2 = m(m-1)a^2(ax+b)^{m-2}$

$y_3 = m(m-1)(m-2)a^3(ax+b)^{m-3}$

Proceeding in a similar manner we have

$y_n = m(m-1)(m-2)......(m-(n-1))\,a^n(ax+b)^{m-n}$

$\therefore\ D^n = (ax+b)^m = m(m-1)(m-2)......(m-n+1)\,a^n(ax+b)^{m-n}$

Note: If m is a positive integer greater than n

Then since $m(m-)(m-2)....(m-n+1) = \dfrac{m!}{(m-n)!}$

$$\therefore\ D^n(ax+b)^m = \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$$

Note: If m is a positive integer less than n.

Then $D^n(ax+b)^n = 0$

When m=n $D^n(ax+b)^n = a^n.n!$

(iii) $y = e^{ax}$

By actual differentiation

$y_1 = ae^{ax},\ y_2 = a^2e^{ax},\ y_3 = a^3e^{ax}$

Proceeding in a similar manner we have

$y_n = a^ne^{ax}$

$\therefore\ D^n(e^{ax}) = a^ne^{ax}$

Cor: $D^n(e^x) = e^x$

Cor: $D^n(a^x) = a^x\left(\log_e^a\right)^n$

(iv) $y = \dfrac{1}{x+a}$

By actual differentiation

$y_1 = -1\ (x+a)^{-2},\ y_2 = (-1)\ (-2)\ (x+a)^{-3} = (-1)^2\ 2!\ (x+a)^{-3}$

$y_3 = (-1)\ (-2)\ (-3)\ (x+a)^{-4} = (-1)^3\ 3!\ (x+a)^{-4}$

Proceeding in a similar manner we have

$y_n = (-1)^n\ n!\ (x+a)^{-(n+1)}$

$$\therefore \quad D^n \left(\frac{1}{x+a}\right) = \frac{(-1)^n n!}{(x+a)^{n-1}}$$

$$\text{Cor: } D^n \left\{\frac{1}{(ax+b)^m}\right\} = \frac{(-1)^n a^n (m+n-1)!}{(m-1)!\,(ax+b)^{m-n}}$$

(v) $y = \log (x+a)$

By actual differentiation

$$y_1 = \frac{1}{x+a},\ y_2 = -1\ (x+a)^{-2},\ y_3 = (-1)\ (-2)\ (x+a)^{-3}$$

$$= (-1)^2\ 2!\ (x+a)^{-3}$$

Hence using (iv) above we have

$D^n (\log (x+a)) = (-1)^{n-1}\ (n-1)!\ (x+a)^{-n}$

$$= \frac{(-1)^{n-1}(n-1)!}{(x+a)^n}$$

$$\text{Cor: } D_n\ \{\log (ax+b)\} = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$$

(vi) $y = \sin (ax + b)$

By actual differentiation

$$y_1 = a \cos (ax+b) = a \sin \left\{\frac{\pi}{2}+(ax+b)\right\}$$

$$y_2 = -a^2 \sin (ax+b) = a^2 \sin \left\{2\frac{\pi}{2}+(ax+b)\right\}$$

$$y_3 = -a^3 \cos (ax+b) = a^3 \sin \left\{3\frac{\pi}{2}+(ax+b)\right\}$$

Proceeding in a similar manner we have

$$y_n = a^n \sin\left\{n\frac{\pi}{2}+(ax+b)\right\}$$

Therefore Dn {sin(ax + b)} $= a^n \sin\left\{n\frac{\pi}{2}+(ax+b)\right\}$

(vii) $y = \cos(ax + b)$

By actual differentiation

$$y_1 = -a\sin(ax+b) = a\cos\left\{\frac{\pi}{2}+(ax+b)\right\}$$

$$y_2 = -a^2\cos(ax+b) = a^2\cos\left\{2\frac{\pi}{2}+(ax+b)\right\}$$

$$y_3 = a^3\sin(ax+b) = a^3\cos\left\{3\frac{\pi}{2}+(ax+b)\right\}$$

Proceeding in a similar manner we have

$$y_n = a^n\cos\left\{n\frac{\pi}{2}+(ax+b)\right\}$$

$$\therefore \quad D^n(\cos(ax+b)) = a^n\cos\left\{n\frac{\pi}{2}+(ax+b)\right\}$$

Illustrative Examples

Example 1: If $y = \sin^3 x$, find y_n

Solution: We know that $\sin 3x = 3\sin x - 4\sin^3 x$

$$y = \sin^3 x = \frac{1}{4}(3\sin x - \sin 3x)$$

By actual differentiation and using (vi) we have

$$y_n = \frac{1}{4}\left[3\sin\left(n\frac{\pi}{2}+x\right) - 3^n\sin\left(n\frac{\pi}{2}+3x\right)\right]$$

$$\text{i.e } D^n(\sin^3 x) = \frac{1}{4}\left[3\sin\left(n\frac{\pi}{2}+x\right) - 3^n\sin\left(n\frac{\pi}{2}+3x\right)\right]$$

Example 2: If $y = e^{ax}$ sinbx, find y_n

By actual differentiation

$y_1 = ae^{ax} \sin bx + be^{ax} \cos bx$

$= e^{ax} (a \sin bx + b \cos bx)$

Putting $a = r \cos\phi$ and $b = r \sin\phi$ so that $r^2 = a^2 + b^2$ and $\tan\phi = \frac{b}{a}$ we have

$y_1 = e^{ax} (r \cos\phi \sin bx + r \sin\phi \cos bx)$

$= r e^{ax} \sin (bx + \phi)$

Similarly $y_2 = r e^{ax} [a \sin (bx+\phi) + b \cos (bx+\phi)$

$= r^2 e^{ax} \sin (bx+2\phi)$

Procceding in a similar manner we have

$y_n = r^n e^{ax} \sin (bx + n\phi)$

i.e. $D^n (e^{ax} \sin bx) = r^n e^{ax} \sin (bx + n\phi)$

Cor : $D^n(e^{ax} \cos bx) = r^n e^{ax} \cos (bx+n\phi)$

Example 3. If $y = \sin (m \sin^{-1}x)$, Show that

$(1 - x^2) y_{n+2} - (2n + 1)xy_{n+1} + (n^2 - m^2) y_n = 0$ (NEHU 2007)

Solution : $y = \sin (m \sin^{-1}x)$ ..... (1)

By Actual diffentiation we get

$$y_1 = \cos (m \sin^{-1}x) \left(m \frac{1}{\sqrt{1-x^2}}\right)$$

i.e $$y_1^2 = \cos^2 (m \sin^{-1}x) \left(\frac{m^2}{1-x^2}\right)$$

i.e $y_1^2 (1-x^2) = m^2 \cos^2 (m \sin^{-1}x)$

$\Rightarrow$ $y_1^2 (1-x^2) = m^2 [1 - \sin^2 (m \sin^{-1}x)]$

$\Rightarrow$ $y_1^2 (1-x^2) = m^2 [1 - y^2]$ (i)

$\Rightarrow$ $y_1^2 (1-x^2) + m^2y^2 = m^2$ ....(A)

Differentiating (A) w.r.t x we have

$2y_1 (1-x^2) y_2 + y_1^2 (-2x) + m^2 2y y_1 = 0$

$\Rightarrow$ $2 (1-x^2) y_2 - 2xy_1 + 2m^2y = 0$

$\Rightarrow \quad (1-x^2)\, y_2 - xy_1 + m^2y = 0$ ....(i)

Differentiating again w.r.t x we have

$$(1-x^2)\, y_3 + y_2(-2x) - xy_2 - y_1 + m^2y_1 = 0$$

$\Rightarrow \quad (1-x^2)\, y_3 - 3xy_2 - (1-m)^2\, y_1 = 0$

i.e $\quad (1-x^2)\, y_{1+2} - (2.1+1)x\, y_{1+1} + (1^2-m^2)\, y_1 = 0$

Procceding in the same manner and differentiating successively (A) n times we get

$$(1-x^2)\, y_{n+2} - (2n+1)x\, y_{n+1} + (n^2-m^2)\, y_n = 0$$

Example 4: If $y = \log\left(x+\sqrt{a^2+x^2}\right)$, then show that

$$(a^2 + x^2)\, y + xy_1 = 0$$ (NEHU 2013)

Solution: $y = \log\left(x+\sqrt{a^2+x^2}\right)$ .... (i)

By actual differentiation

$$y_1 = \frac{1}{x+\sqrt{a^2+x^2}}\left(1+\frac{2x}{2\sqrt{a^2+x^2}}\right)$$

i.e $\quad y_1 = \dfrac{1}{x+\sqrt{a^2+x^2}} \times \dfrac{x+\sqrt{a^2+x^2}}{\sqrt{a^2+x^2}}$

i.e $\quad y_1 = \dfrac{1}{\sqrt{a^2+x^2}}$

$\Rightarrow \quad y_1^2\, (a^2 + x^2) = 1$ ....(A)

Diff (A) w.r.t x we get

$$2y_1y_2\, (a^2 + x^2) + y_1^2.2x = 0$$

i.e $\quad y_2\, (a^2 + x^2) + xy_1 = 0$

Example 5: If $y = \dfrac{1}{ax+b}$, find y_n (NEHU 2016)

Solution: Given $y = \dfrac{1}{ax+b} = (ax + b)^{-1}$

By actual differentiation we get

$y_1 = (-1)\ (ax+b)^{-2}.a$

$y_2 = (-1)\ (-2)^{-3}.a^2 = (-1)^2\ 2!\ (ax+b)^{-3}.a^2$

$= (-1)\ (-2)\ (ax+b)^{-3}.a^2$

$y_3 = (-1)\ (-2)\ (-3)\ (ax+b)^{-4}\ a^3 = (-1)^3\ 3!\ (ax+b)^{-4}.a^3$

Proceeding in the same manner and differntiating successively n times we get

$y_n = (-1)^n\ n!\ (ax+b)^{-(n+1)}.a^n$

i.e $y_n = \dfrac{(-1)^n\, a^n.n!}{(ax+b)^{n-1}}$

Example 6: If $\log y = \tan^{-1}x$, then prove that

(i) $(1 + x^2)y_2 + (2x - 1)\ y_1 = 0$

(ii) $(1 + x^2)y_{n+2} + (2nx + 2x - 1)\ y_{n+1} + n\ (n+1)\ y_n = 0$

(NEHU 2010 2016)

Solution: Given $\log y = \tan^{-1}x$

By actual differentiation we get

$$\frac{1}{y}\ y_1 = \frac{1}{1+x^2}$$

i.e $(1+x^2)\ y_1 = y$ ....(1)

Differentiating again w.r.t x we get

$(1+x^2)\ y_2 + y_1.2x = y_1$

i.e $(1+x^2)\ y_2 + (2x-1)\ y_1 = 0$ ....(A)

Differentiating (A) w.r.t x we get

$(1+x^2)\ y_3 + y_2\,.\ 2x + (2x-1)\ y_2 + 2y_1 = 0$

i.e $(1+x^2)\ y_3 + (2x+2x-1)y_2 + 2y_1 = 0$ ......(ii)

i.e $(1+x^2)\ y_{1+2} + (2.1.x+2.x-1)y_{1+1} + 1(1+1)y_1 = 0$

Differentiating (ii) again w.r.t x we get

$(1+x^2)\ y_4 + y_3.2x + (4x-1)y_3 + 4y_2 + 2y_2 = 0$

i.e $(1+x^2)\ y_4 + (4x+2x-1)y_3 + 6y_2 = 0$

i.e $(1+x^2)\ y_{2+2} + (2.2x+2x-1)y_{2+1} + 2(2+1)y_2 = 0$

Hence proceeding in the same manner and differentiating (A) w.r.t x n times we get

$(1+x^2)\ y_{n+2} + (2nx+2x-1)y_{n+1} + n(n+1)y_n = 0$

Example 7: If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$, $|x| < 1$, then show that

(i) $(1-x^2)\, y_2 - 3xy_1 - y = 0$

(ii) $(1-x^2)\, y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2 y_n = 0$ (NEHU 2014)

Solution: Given $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$ ....(i)

i.e $y^2\,(1-x^2) = (sn^{-1}x)^2$

By actual differentiation we get

$$2yy_1\,(1-x^2) + y^2\,(-2x) = 2(\sin^{-1}x)\,\frac{1}{\sqrt{1-x^2}}$$

i.e $2yy_1\,(1-x^2) - 2xy^2 = 2y$ by (i)

i.e $(1-x^2)y_1 - xy = 2$ ....(ii)

differentiating again w.r.t x we get

$$(1-x^2)\, y_2 + y_1(-2x) - xy_1 - y = 0$$

i.e $(1-x^2)\, y_2 - 3xy_1 - y = 0$ ....(A)

Differentiating (A) w.r.t. x we get

$$(1-x^2)\, y_3 - 2xy_2 - 3xy_2 - 3y_1 - y_1 = 0$$

i.e $(1-x^2)\, y_3 - 5xy_2 - 4y_1 = 0$ ....(iii)

i.e $(1-x^2)\, y_{1+2} - (2.1+3)xy_{1+1} - (1+1)^2\, y_1 = 0$

Differentiating (iii) again w.r.t x we get

$$(1-x^2)\, y_4 - 2xy_3 - 5xy_2 - 4y_2 = 0$$

i.e $(1-x^2)\, y_4 - 7xy_3 - 9y_2 = 0$

i.e $(1-x^2)\, y_{2+2} - (2.2+3)xy_{2+1} - (2+1)^2\, y_2 = 0$

Proceeding in the same manner and by differentiating (A) successively w.r.t x n times we get

$$(1-x^2)\, y_{n+2} - (2n+3)xy_{n+1} - (n+1)^2\, y_n = 0$$

Example 8: If $y = \tan^{-1}x$, show that

$$(1+x^2)\, y_{n+1} + 2nxy_n + n(n-1)\, y_{n-1} = 0$$

Also find $(y_4)_0$ (NEHU 2006, 2016)

Solution: Given $y = \tan^{-1}x$

By actual differentiation

$y_1 = \frac{1}{1+x^2}$

i.e y_1 $(1+x^2) = 0$ ....(A)

Differentiating (A) again w.r.t x we get

$y_2(1+x^2) + 2xy_1 = 0$ ....(ii)

i.e $(1+x^2)y_{1+1} + 2.1xy_1 + 1(1-1)y_{1-1} = 0$

Differentiating (ii) again w.r.t x we get

$y_3(1+x^2) + 2xy_2 + 2xy_2 + 2y_1 = 0$

i.e $(1+x^2)y_3 + 4xy_2 + 2y_1 = 0$ ....(iii)

i.e $(1+x^2)y_{2+1} + 2.2xy_2 + 2(2-1)y_{2-1} = 0$

Differentiating (iii) again w.r.t x we get

$(1+x^2)y_4 + 2xy_3 + 4xy_3 + 4y_2 + 2y_2 = 0$

i.e $(1+x^2)y_4 + 6xy_3 + 6y_2 = 0$ ....(iv)

i.e $(1+x^2)y_{3+1} + 2.3xy_3 + 3(3-1)y_{3-1} = 0$

Hence by differentiating (A) successively n time w.r.t x we get

$(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$

From (iv) when x = 0

$y_4 = -6y_2$

and from (ii) when x=0, y=0

Hence y_4=0

Example 9: If $y = x^{2n}$ where n is a positive integer. Show that $y_n = 2^n\{1.3.5....(2n-1)\}x^n$ (NEHU 2002)

Solution: Given $y = x^{2n}$

By actual differentiation we have

$y_1 = 2n\, x^{2n-1}$, $y_2 = 2n\,(2n-1)x^{2n-2}$

$y_3 = 2n\,(2n-1)\,(2n-2)\,x^{2n-3}$

Proceeding in the same manner we have

$y_n = 2n(2n-1)\,(2n-2)\,.....\,(2n-(n-1))\,x^{2n-n}$

$= 2n(2n-1)\,(2n-2)\,.....\,(n+1)\,x^n$

$= \frac{(2n)!}{n!}x^n$

7.3. Successive differentiation of product of two differentiable functions

Leibnitz's Theorem

If u and v are two functions of x possessing derevative of n^{th} order then

$$(uv)_n = {}^nc_0 u_n v + {}^nc_1 y_{n-1} u_1 + {}^nc_2 u_{n-2} v_2 + ... + {}^nc_r u_{n-r} v_r + ... + {}^nc_n u v_n$$

where the suffixes of u an v denote the order of differentations of u and v with respect to x.

Proof :

Let y = uv

Then by actual differentiation, we have

$$y_1 = u_1 v + u v_1$$

$$y_2 = u_2 v + 2u_1 v_1 + u v_2 = u v_2$$

i.e $$y_2 = u_2 v + 2u_1 v_1 + u v_2 = u_2 v + {}^2c_1 u_{2-1} v_1 + {}^2c_2 u v_2$$

Again $$y_3 = u_3 v + u_2 v_1 + 2u_2 v_1 + 2u_1 v_2 + u_1 v_2 + u v_3$$
$$= u_3 v + 3u_2 v_1 + 3u_1 v_2 + u v_3$$

i.e $$y_3 = u_3 v + {}^3c_1 u_{3-1} v_1 + {}^3c_2 u_{3-2} v_2 + {}^3c_3 u v_3$$

Thus the theorem is true for n = 1, 2

We assume that the theorem is true for any possivive integral value of n say k i.e n = k (k < n)

Then

$$y_k = (uv)_k = u_k v + {}^kc_1\, u_{k-1} v_1 + {}^kc_2\, u_{k-2} v_2 + ... + {}^kc_r\, u_{k-r} v_r + ... + {}^kc_k\, u v_k \quad ..... (A)$$

Differentiating both sides with respect to x again we get

$$y_{k+1} = (uv)_{k+1} = u_{k+1} v + u_k v_1 + {}^kc_1\,(u_{k-1} v_2 + u_k v_1) + {}^kc_2\,(u_{k-1} v_2 + u_{k-2} v_3) + ... + {}^kc_r\,(u_{k-(r+1)} v_2 + u_{k-r} v_{r+1}) + ... + {}^kc_k\,(u_1 v_k + u v_{k+1})$$

i.e $$y_{k+1} = (uv)_{k+1} = u_{k+1} v + (u_k v_1 + {}^kc_1 u_k v_1) + ({}^kc_1 u_{k-1} v_2 + {}^kc_2\, u_{k-1} v_2) + ({}^kc_2\, u_{k-2} v_3 + {}^kc_3 u_{k-2} v_3) + ... + ({}^kc_{k-1}\, u_1 v_k + {}^kc_k u_1 v_k) + {}^kc_k\, u v_{k+1}$$

i.e $$y_{k+1} = u_{k+1} v + (1 + {}^kc_1) u_k v_1 + ({}^kc_1 + {}^kc_2)\, u_{k-1} v_2 + ({}^kc_2 + {}^kc_3)\, u_{k-2} v_3 + ... + ({}^kc_{k-1} + {}^kc_k)\, u_1 v_k + {}^kc_k\, u v_{k+1} \quad ....(i)$$

But ${}^{k}c_0 = {}^{k+1}c_0 = 1$

$1 + {}^{k}c_1 = {}^{k+1}c_1$ and ${}^{k}c_r + {}^{k}c_{r+1} = {}^{k+1}c_r$; ${}^{k}c_k = {}^{k+1}c_{k+1}$

Hence

$$y_{k+1} = (uv)_{k+1} = {}^{k+1}c_0\, u_{k+1}v + {}^{k+1}c_1\, u_k v_1 + {}^{k+1}c_2\, u_{k-1}v_2 + {}^{k+1}c_3\, u_{k-2}v_3 + ... + {}^{k+1}c_k\, u_l v_k + {}^{k+1}c_{k+l}\, uv_{k+l}$$

i.e $$y_{k+1} = (uv)_{k+1} = {}^{k+1}c_0\, u_{k+1}v + {}^{k+1}c_1\, u_{(k+1)-1}\, v_1 + {}^{k+1}c_2\, u_{(k+1)-2}v_2 + {}^{k+1}c_3\, u_{(k+1)-3}v_3 + {}^{k+1}c_4\, u_{(k+1)-4}v_4 + ... + {}^{k+1}c_k\, u_{(k+1)-k}\, v_k + {}^{k+1}c_{k+1}\, uv_{k+1}$$

$$\therefore \quad y_n = (uv)_n = {}^{n}c_0\, u_n v + {}^{n}c_1\, u_{n-1}\, v_1 + {}^{n}c_2\, u_{n-2}\, v_2 + {}^{n}c_3\, u_{n-3}v_3 + ... + {}^{n}c_k\, u_{n-k}\, v_k + ... {}^{n}c_n\, uv_n$$

Thus the theorem is true for any integral value n=k+1 and by Principle of Mathematical Induction, it is true for every positive integral value of n.

7.4 Important Results of Symbolic Operation

If F(D) is any rational integral algebraic function of D or $\frac{d}{dx}$ (the symbolic operator) i.e. if

$$F(D) = A_n D^n + A_{n-1} D^{n-1} + ... + A_1 D + A = \Sigma A_r D^r$$

where A_r is independent of D, then

(i) $F(D)\, e^{ax} = F(a)\, e^{ax}$

(i) $F(D)\, e^{ax}V = e^{ax}\, F(D+a)\, V$, V being function of x

(iii) $$F(D^2) \begin{cases} \sin(ax+b) \\ \cos(ax+b) \end{cases} = F(-a)^2 \begin{cases} \sin(ax+b) \\ \cos(ax+b) \end{cases}$$

Proof:

(i) Since $D^r e^{ax} = a^r e^{ax}$

$$\therefore \quad F(D)\, e^{ax} = \Sigma A_r D^r (e^{ax}) = \Sigma A_r\, a^r\, e^{ax} = \Sigma (A_r\, a^r)\, e^{ax} = F\,(a)\, e^{ax}$$

(ii) Let $y = e^{ax}V$

Since $D^r\, e^{ax} = a^r e^{ax}$

$\therefore$ by Leibnitz's theorem we have

$$y_n = e^{ax}\,(a^n V + {}^{n}c_1\, a^{n-1}\, DV + {}^{n}c_2\, a^{n-2}\, D^2V + ... + D^nV)$$

$\therefore \quad D^n (e^{ax}V) = e^{ax} (D+a)^n V$

$\therefore \quad F(D)\, e^{ax}V = (\Sigma A_r D^r)\, e^{ax} V$

$= \Sigma A_r D^r e^{ax} V$

$= e^{ax} \Sigma A_r (D+a)^r V$

$= e^{ax} F (D+a) V$

(iii) We have $D \sin (ax+b) = a \cos (ax+b)$

So $D^2 \sin (ax+b) = (-a^2) \sin (ax+b)$

$\therefore \quad D^{2r} \sin (ax+b) = (-a^2)^r \sin (ax+b)$

Hence as in (i) and (ii)

$F(D^2) \sin (ax+b) = F(-a^2) \sin (ax+b)$

and $F(D^2) \cos (ax+b) = F(-a^2) \cos (ax+b)$

Illustrative Examples

Example 1: If $y = e^{ax} x^3$, find y_n

Solution: Let $u = e^{ax}$, $v = x^3$

Then $u_1 = ae^x$, $u_2 = a^2 e^{ax}, \ldots\ldots\ldots u_n = a^n e^{ax}$

$v_1 = 3x^2$, $v_2 = 6x$, $v_3 = 6$, $v_4 = 0$

By Leibnitz's Theorem

$$y_n = (e^{ax} x^3)_n = (uv)_n = {}^nc_0\, u_n v + {}^nc_1\, u_{n-1}v_1 + {}^nc_2\, u_{n-2}v_2 + {}^nc_3\, u_{n-3}v_3 + {}^nc_4\, u_{n-4}v_4$$

$$= {}^nc_0\, a^n e^{ax} x^3 + {}^nc_1\, a^{n-1} e^{ax} 3x^2 + {}^nc_2\, a^{n-2} e^{ax} 6x + {}^nc_3\, a^{n-3}e^{ax}6$$

$$= a^n e^{ax} x^3 + n\, a^{n-1} e^{ax} 3x^2 + \frac{n(n-1)}{2!} a^{n-2} e^{ax} 6x + \frac{n(n-1)(n-2)}{3!} a^{n-3} e^{ax} 6$$

$$= e^{ax} x^{n-3} \{a^3 x^3 + 3na^2x^2 + 3n(n-1)\, ax + n(n-1)(n-2)\}$$

Example 2: Let $y = x^2 e^{ax}$. Compute $\frac{d^n y}{dx^n}$, where n is a positive integer. State the theorem you have used (NEHU 2003)

Solution: Let $u = e^{ax}$, $v = x^2$

Then $u_1 = ae^{ax}$, $u_2 = a^2e^{ax}, \ldots. u_n = a^n e^{ax}$

$v_1 = 2x$, $v_2 = 2_1$, $v_3 = 0$

$\therefore$ By Leibnitz's Theorem we have

$$y_n = (x^2 e^{ax})_n = {}^nc_0\, u_n v + {}^nc_1\, u_{n-1} v_1 + {}^nc_2\, u_{n-2} v_2$$

$$= a^n e^{ax}.x^2 + n.a^{n-1}e^{ax}.2x + \frac{n(n-1)}{2!} a^{n-2} e^{ax}.2$$

$$= a^n e^{ax}.x^2 + 2n.a^{n-1}e^{ax}.x + n(n-1)\, a^{n-2} e^{ax}$$

Example 3: Let $y = x^3 \sin x$. Compute $\frac{d^n y}{dx^n}$, where n is a positive integer. State the theorem you have used. (NEHU 2008)

Solution: Let $u = \sin x$, $v = x^3$

Then $u_1 = \cos x = \sin (\pi/2 + x)$

$u_2 = -\sin x = \sin (2\pi/2 + x)$

$u_n = \sin (n\pi/2 + x)$

$v_1 = 3x^2$, $v_2 = 6x$, $v_3 = 6$, $v_4 = 0$

$\therefore$ By Leibnitz's Theorem we have

$$y_n = (x^3 \sin^x)_n = {}^nc_0\, u_n v + {}^nc_1\, u_{n-1} v_1 + {}^nc_2\, u_{n-2} v_2 + {}^nc_3\, u_{n-3} v_3$$

$$= {}^nc_0 \sin (n\pi/2 + x)\, x^3 + {}^nc_1 \sin \{(n-1)\pi/2 + x\}$$

$$+ {}^nc_2 \sin \{(n-2)\pi/2 + x)\}\, 6x + {}^nc_3 \sin \{(n-3)\pi/2 + x)\}6$$

$$= \sin (n\pi/2 + x)\, x^3 + n \sin \{(n-1)\pi/2 + x\}.3x^2 + \frac{n(n-1)}{2!}$$

$$\sin \{(n-2)\pi/2 + x)\}\, 6x + \frac{n(n-1)(n-2)}{3!}$$

$$\sin\{(n-3)\pi/2 + x)\}6$$

$$= \sin (n\pi/2 + x)\, x^3 + n \sin \{(n-1)\pi/2 + x\}.3x^2 + n(n-1)$$

$$\sin \{(n-2)\pi/2 + x)\}\, 3x + n(n-1)(n-2) \sin\{(n-3)\pi/2 + x)\}$$

Example 4: If $y = x^{n-1} \log x$, prove that $y_n = \frac{(n-1)!}{x}$ (NEHU 2005)

Solution: $y_n = D^n y = D^{n-1}(Dy) = D^{n-1}\{\frac{d}{dx} x^{n-1} \log x\}$

$$= D^{n-1}\{(n-1)x^{n-2}\log x + x^{n-1}\frac{1}{x}\}$$

$$= D^{n-1}\{(n-1)x^{n-2}\log x + x^{n-2}\}$$

$$= (n-1)\, D^{n-1}\{x^{n-2}\log x\} + D^{n-1}\{x^{n-2}\}$$

$$= (n-1)\, D^{n-2}\{\frac{d}{dx}(x^{n-2}\log x)\} + D^{n-1}\{x^{n-2}\}$$

$$= (n-1)\, D^{n-2}\{(n-2)\, x^{n-3}\log x + x^{n-3}\} + D^{n-1}\{x^{n-2}\}$$

$$= (n-1)(n-2)\, D^{n-2}\{x^{n-3}\log x\} + (n-1)\, D^{n-2}\{x^{n-3}\} + D^{n-1}\{x^{n-2}\}$$

$$= (n-1)(n-2)\, D^{n-3}\{\frac{d}{dx} x^{n-3}\log x\} + 0$$

$\because$ If $u = x^n$, $D^n u = n!$ $D^{n+1}u = 0$

$$= (n-1)(n-2)\, D^{n-3}\{(n-3)\, x^{n-4}\log x + x^{n-4}\}$$

$$= (n-1)(n-2)(n-3)\, D^{n-3}\{x^{n-4}\log x\} + (n-1)(n-2)\, D^{n-3}\{x^{n-4}\}$$

$$= (n-1)(n-2)(n-3)\, D^{n-3}\{x^{n-4}\log x\}$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$= (n-1)(n-2)(n-3)(n-4) \ldots\ldots 3.2.1\ D\{x^0 \log x\}$$

$$= (n-1)!\ \frac{d}{dx}\log x = \frac{(n-1)!}{x}$$

Exercises

1. Find y_n in the following cases:

(i) $y = (a-bx)^m$ (ii) $y = \frac{1}{(ax+b)^n}$ (iii) $y = \frac{1}{a-x}$ (iv) $y = \sqrt{x}$

(v) $y = \frac{1}{\sqrt{x}}$ (vi) $y = (2-3x)^n$ (vii) $y = 10^{8-2x}$ (viii) $y = e^x \cos x$

(ix) $y = e^{3x}\sin 4x$ (x) $y = e^x\sin^2 x$

2. If $y = x^{2n}$, where n is a positive integer, show that $y_n = 2^n \{1.3.5.....(2n-1)\}x^n$

3. Apply Leibnitz's Theorem to find

   (i) y_2 when $y = x^2 \log x$ (ii) y_3 when $y = x^3\cos x$

   (iii) y_4 when $y = e^x x^3$ (iv) y_4 if $y = e^{-x} \cos x$

   (v) u_n when $u = xy_1$ (vi) v_n when $v = (1+x^2)y_2$

4. Find y_n by Leibnitz's Theorem in the following cases:

   (i) $y = x^3e^{ax}$ (ii) $y = x^3\sin x$ (iii) $y = e^{ax} \cos bx$

   (iv) $y = e^x\log x$ (v) $y = x^2\tan^{-1}x$ (vi) $y = x^3\log x$

5. If $y = A \sin mx + B \cos mx$, then prove that $y_2 + m^2y = 0$

6. If $y = e^{ax} \sin bx$, then show that $y_2 - 2ay_1 + (a^2 + b^2)y = 0$

7. If $y = \left(x+\sqrt{1+x^2}\right)^m$, then prove that $(1+x^2)y_2 + xy_1 - m^2y = 0$

8. If $y = \tan^{-1}x$, show that

   (i) $(1+x^2)y_1 = 0$ (ii) $(1+x^2)y_2 + 2xy_1 = 0$

   (iii) $(1+x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$

   Also find $(y_n)_0$

9. If $y = \sin^{-1}x$, prove that

   (i) $(1-x^2)y_2 - xy_1 = 0$

   (ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$

   Also find yn when x=0

10. If $y = (\sin^{-1} x)^2$, prove that

    (i) $(1-x^2)y_2 - xy_1 = 2$

    (ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$

11. If $y = a \cos(\log x) + b \sin(\log x)$ for x>0, show that

    (i) $x^2y_2 + xy_1 + y = 0$

    (ii) $x^2y_{n+2} + (2x+1)xy_{n+1} + (n^2+1)y_n = 0$

12. If $y = \sin(m \sin^{-1}x)$, show that

    (i) $(1-x^2)y_2 - xy_1 + m^2y = 0$

    (ii) $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2-m^2)y_n = 0$

13. If $y = \cos(m \sin^{-1}x)$ show that

    (i) $(1-x^2)\, y_2 - xy_1 + m^2y = 0$

    (ii) $(1-x^2)y_{n+2} - (2n+1)\, xy_{n+1} + (m^2-n^2)\, y_n = 0$

    Also find the value $(y_n)_0$

14. If $y = \cos(10\cos^{-1}x)$ show that $(1-x^2)y_{12} = 21xy_n$

15. If $y = e^m \sin^{-1}x$, show that

    (i) $(1-x^2)\, y_2 - xy_1 - m^2y = 0$

    (ii) $(1-x^2)y_{n+2} - (2n+1)\, xy_{n+1} - (n^2+m^2)\, y_n = 0$

16. If $y = (x^2-1)^n$, show that

    $(x^2-1)\, y_{n+2} + 2xy_{n+1} - n(n+1)\, y_n = 0$

17. If $y = x^{n-1}\log x$, then show that $y_n = \dfrac{(n-1)!}{x}$

18. If $y = \dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$, $|x|<1$ show that

    (i) $(1-x^2)\, y_2 - 3xy_1 - y = 0$

    (ii) $(1-x^2)y_{n+2} - (2n+3)\, xy_{n+1} - (n+1)^2\, y_n = 0$

8

Mean Value Theorems and Expansion of Functions

Introduction

In this chapter we shall discuss few very useful theorem which are directly connected with the subsequent development of the subject.

8.1 Rolle's Theorem

If (i) $f(x)$ is a continuous function in the closed interval $a \le x \le b$

(ii) $f'(x)$ exists (i.e $f(x)$ is derivable) in the open interval $a<x<b$

and (iii) $f(a) = f(b)$

then there exists at least one value of x (say ξ) such that $f'(\xi) = 0$ for $a<\xi<b$

Proof: If $f(x)$ is constant throughout the interval $a \le x \le b$ and $f(a) = f(b)$ being true, then evidently $f'(x) = 0$ at every point in the interval.

Since $f(x)$ is continuous in the closed interval $[a, b]$ it must be bounded both above and below in $[a, b]$ and it attains its bounds. Let M, m be the least upper bound and greatest lower bounds of $f(x)$ in $[a, b]$ then

(i) If $m = M$, we have $M=m=f(a)=f(b)=f(x)$ for $a \le x \le b$.

So $f(x)$ is constant throughout in the interval $[a, b]$ and as before in the previous argument $f'(x) = 0$ at every point in the interval.

(ii) If $m \neq M$ then either M or m, if not both must be different from the values $f(a)$, $f(b)$.

Suppose $M \neq f(a) = f(b)$

Now if f(x) attain the value M at $x = \xi$ then a<c<b. Since f(x) is differentiable in a<x<b, it is in particular differentiable at $x = \xi$ i.e $f'(\xi)$ exists.

If $f'(\xi) > 0$ then there exists an open interval $\xi < x < \xi + \delta$ such that for every x in this interval $f(x) > f(\xi) = M$ which is again absurd.

Hence the only possibility is that $f'(\xi) = 0$

Thus we can definitely specify at least one point where $f'(x) = 0$

Hence the theorem.

8.2 Corolary

If a, b are two roots of the equation f(x) = 0, then the equation f'(x) = 0 will have at least are root between a and b provided

(i) f(x) is continuous in $a \le x \le b$

and (ii) f'(x) exists in a<x<b

If f(x) is a polynomial the conditions (i) and (ii) are evidently satisfied.

8.3 Geometrical Interpretation of Rolle's Theorem

If the graph of y = f(x) has the ordinates at two points A and B equal, and if the graph is continuous throughout the interval from A and B an if the curve has a tangent at every point on it from A to B except possibly at the two extreme points A and B, then there must exists at least one point on the curve intervening between A and B where the tangent is parellel to x-axis.

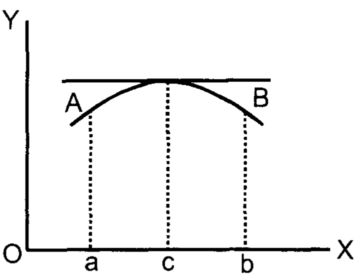


Fig 1

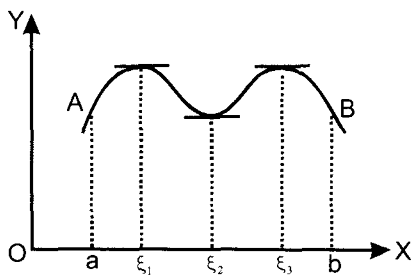


Fig 2

8.4 Remarks

If f(x) statisfies all the conditions of Rolle's Theorem in [a, b], then the conclusion $f'(\xi) = 0$ where $a < \xi < b$ is assured, but if any of the conditions are violated then Rolle's Theorem will not be necessarily be true; it may still be true but the truth is not assured.

In other words, the three conditions of Rolle's Theorem are only a set of sufficient conditions but they are by no means necessary.

The following are some illustrations:

1. Consider $f(x) = x\sqrt{a^2 - x^2}$ in [0, a]

Here

(i) f(x) is continuous in $0 \le x \le a$ i.e [0, a]

(ii) $f'(x) = \dfrac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}$ exists in $0 < x < a$ i.e (0, a)

(iii) $f(0) = f(a) = 0$

All conditions of Rolle's Theorem are satisfied and as such there exists an $x = \xi$ where $f'(\xi) = 0$ i.e. $\xi = \dfrac{a}{\sqrt{2}}$ in (0, a)

2. Consider $f(x) = |x|$ in [−1, 1]

Here

(i) f(x) is continuous in $-1 \le x \le 1$ i.e [−1, 1]

(ii) $f'(x) = 1$ in $0 < x \le 1$,

$= -1$ in $-1 \le x < 0$

and hence f'(x) does not exists at x = 0

(iii) $f(1) = f(-1) = 1$

Note that f'(x) does not vanish anywhere in [−1, 1] and as such Rolle's Theorem fails.

The failure is due to the fact that $f(x) = |x|$ is not differentiable in $-1 < x < 1$ all other conditions being satisfied.

3. Consider $f(x) = \dfrac{1}{x} + \dfrac{1}{x-1}$ in [0, 1]

Here

(i) f(x) is continuous in $0 < x < 1$ i.e (0, 1)

[but not in $0 \le x \le 1$ i.e [0, 1]]

(ii) $f'(x) = \dfrac{1}{(1-x)^2} - \dfrac{1}{x^2}$ exists in $0 < x < 1$ i.e (0, 1)

(iii) $f(0) \neq f(1)$ both being undefined.

Thus the condition of Rolle's Theorem do not hold. But yet there exists an $x=\xi$ where $f(\xi) = 0$ (namely $\xi=\frac{1}{2}$) where $0 < \xi < 1$.

8.5 Mean Value Theorem (Lagrange's form)

If

(i) f(x) is a continuous function in the closed interval $a \le x \le b$

(ii) f(x) exists (i.e f(x) is derivable) in the open interval $a < x < b$

Then there exists at least one value of x say ξ such that $f(b) - f(a) = (b-a) f(\xi)$, for $a < \xi < b$

Proof: If f(a) = f(b), then the theorem reduces to Rolle's Theorem.

Suppose $f(a) \neq f(b)$ and consider the function

$F(x) = f(x) + Ax$ where A is a constant to be chosen such that F(a) = F(b)

i.e $f(a) + Aa = f(b) + Ab$

i.e $A(a-b) = f(b) - f(a)$

$$\text{i.e } A = \frac{f(b)-f(a)}{a-b} = -\frac{f(b)-f(a)}{b-a} \quad \text{........(1)}$$

$$F(x) = f(x) - \frac{f(b)-f(a)}{b-a}.x \quad \text{.........(2)}$$

Now f(x) is continuous in $a \le x \le b$ and $\left\{\frac{f(b)-f(a)}{b-a}.x\right\}$ being a polynomial is also continuous in $a \le x \le b$

Since the sum or difference of two continous functions is continuous, it follows from (2) that F(x) is continuous in $a \le x \le b$

Again, f(x) is differentiable in $a < x < b$ and the polynomial function $\left\{\frac{f(b)-f(a)}{b-a}.x\right\}$ is differntiable in $a < x < b$, it follows that F(x) is differentiable in $a < x < b$

Hence we see that

(i) F(x) is a continous funtion in $a \le x \le b$

(ii) F(x) is differentiable in $a < x < b$ and

(iii) F(a) = F(b)

Thus all condition of Rolle's Theorem are satisfied by F(x) and also there exists at least one value of x say ξ where $a < \xi < b$ such that $f(\xi) = 0$

$$\text{i.e } f'(\xi) - \frac{f(b) - f(a)}{b - a} = 0 \quad \text{by (2)}$$

$$\text{i.e } f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

$$\text{i.e } f(b) - f(a) = (b-a)\, f'(\xi)$$

8.6 Geometrical Interpretation of Mean Value Theorem (Lagrange's form)

If the graph of $y = f(x)$ is continuous throughout the interval [a, b] and if the curve has a tangent at every point in the interval a to b except possibly at the two extreme points a and b, then by Lagrange's Mean Value Theorem.

$$f(\xi) = \frac{f(b) - f(a)}{b - a}$$

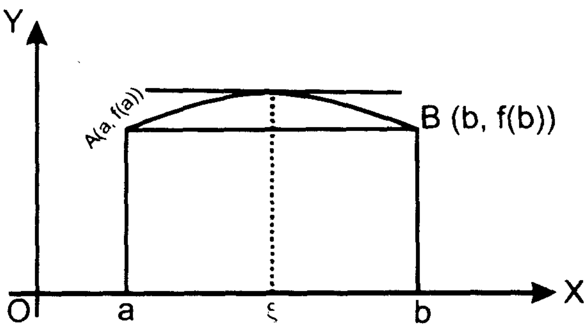
for some point ξ, where $a < \xi < b$

Now if we draw the curve of $y = f(x)$ and take the extreme point A (a, f(a)) and B (b, f(b)) on the curve, then

$$\text{Slope of the Chord AB} = \frac{f(b) - f(a)}{b - a}$$

and thus $f(\xi)$ = slope of the chord AB

Thus the tangent to the curve $y = f(x)$ at the point $x = \xi$ is parallel to the chord AB



8.7 Physical Significance of Mean Value Theorem (Lagrange Form)

Let a particle be moving in a straight line and let f(a) and f(b) be its position from the starting point at times a and b respectively then average speed of the particle $= \dfrac{f(b)-f(a)}{b-a}$

As $f'(\xi)$ is the instantaneous speed of the particle at time ξ and by Mean Value Theorem

$$f'(\xi) = \frac{f(b)-f(a)}{b-a}$$

Hence, Mean Value Theorem says that at some time ξ between a and b, the instantaneous speed of the particle is equal to the average speed.

8.8 Alternative Forms of Lagrange's Mean Value Theorem

(i) Putting $b = a + h$, then any point ξ between a and b can be written as $\xi = a + \theta h$, $0 < \theta < 1$ and consequently Lagrange's mean value theorem takes the form

$f(a+h) - f(a) = h\, f'(a+\theta h)$ for $0 < \theta < 1$

(ii) Putting $a = h$ and $b = h + x$, then any point ξ between a and b can be written as $\xi = h + \theta x$ where $0 < \theta < 1$ and Lagrange's Mean Value Theorem can be stated as

If (a) f(x) is continuous in the closed interval [h, h+x]

(b) f'(x) exists (i.e f(x) is derivable)in the open interval]h, h+x[

Then $f(h+x) - f(h) = x\, f'(h + \theta x)$, $0 < \theta < 1$

(iii) Putting h=0 in (ii) above we obtain Maclaurin's Formula which is

$f(x) = f(0) + x\, f'(\theta x)$, $0 < \theta < 1$

8.9 Important Conclusion from Lagrange's Mean Value Theorem

Corollary 1: If $f'(x) = 0$ in the interval $a \le x \le b$ then f(x) is constant in this interval

Proof: Let $a \le x_1 \le x_2 \le b$

Since f(x) satisfies the condition of Mean Value Theorem in $x_1 \le x \le x_2$, then

$f(x_2) - f(x_1) = (x_2 - x_1)\, f'(\xi)$ where $x_1 < \xi < x_2$

i.e $f(x_2) - f(x_1) = 0 \quad \because f'(\xi) = 0$

i.e $f(x_2) = f(x_1)$

Thus $f(x)$ has the same value at all points between a and b and hence $f(x)$ is a constant

Corollary 2: If $f'(x) = 0$ in $a \le x \le b$, then $f(x) = f(a)$ in $a \le x \le b$

Corollary 3: If $f'(x) = g'(x)$ in the interval $a \le x \le b$, then $f(x) - g(x) =$ constant in $a \le x \le b$

Proof: Consider a function

$\phi(x) = f(x) - g(x)$ in $a \le x \le b$

Then $\phi'(x) = f'(x) - g'(x)$

i.e $\phi'(x) = 0$ in $a \le x \le b$ since $f'(x) = g'(x)$

Hence by corollary 1, $\phi(x)$ is constant in $a \le x \le b$

i.e $f(x) - g(x) =$ constant

Corollary 4: If $f(x)$ is continuous an $a \le x \le b$ and $f'(x) > 0$ in $a < x < b$, then $f(x)$ is strictly increasing function in $a \le x \le b$

Proof: Let x_1 and x_2 be such that $a \le x_1 \le x_2 \le b$

Then by mean value theorem (Lagrange's form)

$f(x_2) - f(x_1) = (x_2 - x_1) f'(\xi)$ where $x_1 \le \xi \le x_2$

i.e $f(x_2) - f(x_1) > 0$ since $f'(\xi) > 0$

i.e $f(x_2) > f(x_1)$

Hence we see that if $x_1 < x_2$ then $f(x_1) < f(x_2)$

Showing that $f(x)$ is a strictly increasing function.

Corollary 5: If $f(x)$ is continuous in $a \le x \le b$ and $f'(x) < 0$ in $a < x < b$, then $f(x)$ is a strictly decreasing function in $a \le x \le b$

Proof: Similar to corollary 4

8.10 Mean Value Theorem (Cauchy's Form)

If

(i) $f(x)$ and $g(x)$ be both continuous in $a \le x \le b$

(ii) $f'(x)$ and $g'(x)$ both exists i.e $f(x)$ and $g(x)$ both are derivable in $a < x < b$

(iii) $g'(x)$ does not vanish at any value of x in the open interval $a < x < b$

Then there exists at least one value of x say ξ such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f\ (\xi)}{g\ (\xi)} \text{ for } a < \xi < b$$

Proof: Consider a function

F(x) = f(x) + A g(x) where A is a constant to be chosen such that F(a) = F(b)

i.e $f(a) + A\, g(a) = f(b) + A\, g(b)$

i.e $A\, (g(a) - g(b)) = f(b) + f(b)$

i.e $A = \dfrac{f(b)-f(a)}{g(a)-g(b)}$

$\therefore \quad F(x) = f(x) + \dfrac{f(b)-f(a)}{g(a)-g(b)} \cdot g(x)$ ....(1)

Now f(x) and g(x) are continuous on $a \le x \le b$ hence F(x) is continuous in $a \le x \le b$

Also f(x) and g(x) are both differentiable derivable in $a < x < b$ hence F(x) is derivable in $a < x < b$

Thus we see that

(i) F(x) is continous in $a \le x \le b$

(ii) F(x) is derivable in $a < x < b$ and

(iii) F(a) = F(b)

Thus all the conditions of Rolle's Theorem is satisfied by F(x) and so there exists at least one value of x say ξ where $a < \xi < b$ such that $F(\xi) = 0$

i.e $f(\xi) + \dfrac{f(b)-f(a)}{g(a)-g(b)}\, g(\xi) = 0$

i.e $\dfrac{f(b)-f(a)}{g(a)-g(b)}\, g(\xi) = -\, f(\xi)$

i.e $\dfrac{f(b)-f(a)}{g(a)-g(b)} = -\, \dfrac{f\ (\xi)}{g\ (\xi)}$

i.e $\dfrac{f(b)-f(a)}{g(a)-g(b)} = \dfrac{f\ (\xi)}{g\ (\xi)}$

Note: In the above theorem $g(b) - g(a) \neq 0$

For if $g(b) - g(a) = 0$ i.e $g(b) = g(a)$

Then g(x) being continous in $a \leq x \leq b$

and g(x) being derivable in $a < x < b$

Then by Rolle's Theorem there exist at least one value of x say ξ such that $g'(\xi) = 0$ where $a < \xi < b$

But $g'(\xi) = 0$ contradicts the hypothesis that $g'(x)$ does not vanish at any value of x in $a < x < b$.

Hence our assumption that $g(b) - g(a) = o$ is wrong and therefore $g(b) - g(a) \neq 0$.

Illustrative Examples

Example 1: If f(x) = tanx, then f(x) vanishes for x = 0 and x = π. Is Rolle's Theorem applicable to the function f(x) in $[0, \pi]$? Give justifications for your answer.

Solution: We see that $f'(x) = \sec^2 x$ does not vanish for any value of x between 0 and π.

Hence Rolle's Theorem is not applicable

Also note that $f'(x) = \sec^2 x$ exists in $0 < x < \pi$ except at $x = \pi/2$ and f(x) is continuous in $[0, \pi]$ except at $x = \pi/2$

Hence the conditions under which Rolle's Theorem is valid do not hold and this explains the failure of the theorem.

Example 2: Is Mean Value Theorem valid for $f(x) = x^2 + 3x + 2$ in $1 \leq x \leq 2$? Find ξ if the theorem is applicable

Solution: Clearly $f(x) = x^2 + 3x + 2$ being a polynomial is continuous in [1, 2]

Also $f'(x) = 2x + 3$ exists in $1 < x < 2$

Hence MVT is applicable

So using Lagrange's MVT we have

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}$$

$$\text{i.e } f'(\xi) = \frac{f(2) - f(1)}{2 - 1} = \frac{12 - 6}{1} = 6$$

$$\text{i.e } 2\xi + 3 = 6 \Rightarrow 2\xi = 3 \Rightarrow \xi = \frac{3}{2} \in [1, 2]$$

Example 3: Use MVT in making numerical approximations to $(28)^{1/3}$

Solution: We have f(x+h) = f(x) + hf′ (x+θh), 0<θ<1

Putting x = 27, h = 1 and f(x) = $x^{1/3}$ we get

$$28^{1/3} = (27)^{1/3} + 1.\frac{1}{3(27+\theta)^{2/3}}$$

which is $< 3 + \frac{1}{3(27)^{2/3}}$ and > 3

$$\therefore 3 < (28)^{1/3} < 3 + \frac{1}{27}$$

Example 4: Show that x > sinx for $0 < x < \pi/2$

Solution: Let f(x) = x – sinx

Then f′(x) = 1 – cosx > 0 in $0 < x < \pi/2$

Also f(x) = 0 at x = 0

Thus f(x) = x – sinx is an increasing function and > 0

Hence x > sinx

Example 5: Find a point on the parabola $y = (x-3)^2$ where the tangent is parrallel to the chord joining (3, 0) and (4, 1)

Solution: We apply Lagrange's MVT for the function

$f(x) = y = (x-3)^2$ in the interval [3, 4]

Now f(x) being a polynomial function, it is continuous in [3, 4]

Also f′(x) = 2(x–3) exists in (3, 4)

Thus both the conditions of Lagrange's MVT hold in this case, hence there exists a point c ∈ [3, 4] such that

$$f'(c) = \frac{f(4)-f(3)}{4-3} \quad 1-0 = 1$$

i.e $2(c-3) = 1 \Rightarrow c-3 = \frac{1}{2} \Rightarrow c = \frac{1}{2}+3 = \frac{7}{2} \in [3, 4]$

Now at $x = \frac{7}{2}$, $y = \left(\frac{7}{2}-3\right)^2 = \frac{1}{4}$

Thus at the point $\left(\frac{7}{2}, \frac{1}{4}\right)$ on the given curve the tangent is parallel to the chord forming (3, 0) and (4, 1)

Example 6: If $f(x) = (x-a)^m (x-b)^n$ where m and n are positgive integers, show that c in Rolle's Theorem divides the segment $a \le x \le b$ in the ratio m:n (NEHU 2008)

Solution: Clearly $f(x) = (x-a)^m (x-b)^n$ is a continuous function in $a \le x \le b$ and $f(a) = 0 = f(b)$

Also $f^1(x) = m(x-a)^{m-1} (x-b)^n + n(x-a)^m (x-b)^{n-1}$ exist for every value of x in $a \le x \le b$

Hence by Rolle's Theorem, there exists $c \in [a, b]$ such that $f'(c) = 0$

i.e $m(c-a)^{m-1} (c-b)^n + n(c-a)^m (c-b)^{n-1} = 0$

or $(c-a)^{m-1} (c-b)^{n-1} [m(c-b) + n(c-a) = 0$

or $m(c-b) + n(c-a) = 0$ $\quad \because a < c < b$

or $mc - mb + nc - na = 0$

or $c = \dfrac{na+mb}{m+n}$

Showing that c divides the segment $a \le x \le b$ in m:n

8.11 Taylor's Theorem in Lagrange's Form of Remainder

If f(x) possesses differential coefficients of the first (n–1) order for every value of x in the closed interval $a \le x \le b$ and the nth derivative of f(x) exists in the open interval $a < x < b$ i.e $f^{n-1}(x)$ is continuous in $a \le x \le b$ and $f^n(x)$ exists in $a < x < b$ then

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f^2(a) + \ldots\ldots +$$

$$\frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)^n}{n!} f^n(\xi) \text{ where } a < \xi < b$$

If $b = a+h$, so that $b-a=h$, then

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f^2(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a)$$

$$+ \frac{h^n}{n!} f^n(a + \theta h) \text{ where } 0 < \theta < 1$$

If a = x, then

$$f(x+h) = f(x) + h\, f'(x) + \frac{h^2}{2!}\, f^2(x) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!}\, f^{n-1}(x)$$

$$+ \frac{h^n}{n!}\, f^n(x + \theta h) \text{ where } 0 < \theta < 1$$

Proof: We observe that f(x), $f^1(x)$, $f^2(x)$ ....$f^{n-1}(x)$ are continuous in $a \le x \le b$

Consider the function $\phi(x)$ defined in (a, b) by

$$\phi(x) = f(b) - f(x) - (b-x)\, f'(x) - \frac{(b-x)^2}{2!}\, f^2(x) \ldots\ldots$$

$$\ldots\ldots \frac{(b-x)^{n-1}}{(n-1)!}\, f^{n-1}(x) - \frac{(b-x)^{n-1}}{(b-a)^n}\, \psi(a)$$

Where $\psi(x) = f(b) - f(x) - (b-x)\, f'(x) \ldots\ldots\ldots \frac{(b-x)^{n-1}}{(n-1)!}\, f^{n-1}(x)$

Then $\phi(a) = \phi(b) = 0$ since $\psi(b) = 0$ identically

Now $\psi'(x) = -f(x) + \{f'(x) - (b-x)\, f^2(x)\} + \{(b-x)\, f^2(x)$

$$-\frac{(b-x)^2}{2!}\, f^3(x)\} + \ldots\ldots + \left\{\frac{(b-x)^{n-2}}{(n-2)!} f^{n-1}(x) - \frac{(b-x)^{n-1}}{(n-1)!} f^n(x)\right\}$$

$$= -\frac{(b-x)^{n-1}}{(n-1)!}\, f^n(x)$$

Hence $\phi'(x) = \frac{(b-x)^{n-1}}{(n-1)!}\, f^n(x) + \frac{n(b-x)^{n-1}}{(b-a)^n}\, \psi(a)$

Since $\phi(a) = \phi(b)$ and $\phi'(x)$ exists in $a < x < b$, by Rolle's Theorem $\phi'(\xi) = 0$ where $a < \xi < b$

i.e $\frac{-(b-\xi)^{n-1}}{(n-1)!}\, f^n(\xi) + \frac{n(b-\xi)^{n-1}}{(b-a)^n}\, \psi(a) = 0$

$$\Rightarrow \quad \psi(a) = \frac{(b-a)^n}{n!} f^n(\xi)$$

and since $\psi(a) = f(b) - f(a) - (b-a) f^1(a) - \frac{(b-a)^2}{2!} f^2(a)$

$$\ldots\ldots\ldots \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a)$$

Hence $f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f^2(a) + \ldots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a)$

$$+ \frac{(b-a)^n}{n!} f^n(\xi)$$

and hence the result follows

Since $a < \xi < b$, we write $\xi = a + (b-a)\theta$

i.e $\xi = a + h\theta$ where $0 < \theta < 1$ and $h = b-a$

and hence

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f^2(a) + \ldots\ldots$$

Note 1. The series

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f^2(a) + \ldots + \frac{(b-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(b-a)^n}{n!} f^n(\xi)$$

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f^2(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a+\theta h)$$

and $f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f^2(x) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(x) + \frac{h^n}{n!} f^n(x+\theta h)$

is called the Taylor's Series with the remainder in Lagrange's form where the remainder after n term being $\frac{(b-a)^n}{n!} f^n(\xi)$, or $\frac{h^n}{n!} f^n(a+\theta h)$ or $\frac{h^n}{n!} f^n(x+\theta h)$ $0 < \theta < 1$ is generally denoted by R_n.

Note 2. Putting n = 1, we get

$f(a+h) = f(a) + h\, f'(a+\theta h)\ 0 < \theta < 1$ which is the Mean Value Theorem.

Remark: Taylor's Theorem is sometimes called the Mean Value Theorem of the n^{th} order.

Note 3. Putting n = 2 we get

$$f(a+h) = f(a) + h\, f'(a) + \frac{h^2}{2!} f^2 (a+\theta h)\ 0 < \theta < 1$$

which is often called the Mean Value Theorem of the Second order and so on.

8.12 Maclaurin's Series in Finite form

Putting x=0, h=x in

$$f(x+h) = f(2) + h\, f'(x) + \frac{h^2}{2!} f^2 (x) + \ldots\ldots + \frac{h^n}{n!} (x+\theta h)$$

We get

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{2!} f^2 (x) + \ldots\ldots + \frac{x^{n-1}}{(n-1)!} f^{n-1}(0)$$

$$+ \frac{x^n}{n!} f^n(\theta x) \quad 0 < \theta < 1$$

Which is called the Madaurin's Series for f(x) with the corresponding remainder R_n being $\frac{x^n}{n!} f^n(\theta x)$

8.13 Cauchy's Form of Remainder R_n

If we take $\phi(x) = \psi(x) - \frac{b-x}{b-a} \psi(a)$

where $\psi(x) = f(b) - f(x) - (b-x)\, f^1(x) \ldots\ldots \frac{(b-x)^{n-1}}{(n-1)!} f^{n-1}(x)$

with $\phi(a) = \phi(b) = 0$, we get as before

$$\phi'(x) = -\frac{(b-x)^{n-1}}{(n-1)!} f^n(n) + \frac{1}{(b-a)} \psi(a) \ldots\ldots(1)$$

Since $\phi(a) = \phi(b)$ and $\phi'(x)$ exists in (a, b) by Rolle's theorem, we have $\phi'(\xi) = 0,\ a < \xi < b$

$$\text{i.e}\quad -\frac{(b-\xi)^{n-1}}{(n-1)!} f^n(\xi) + \frac{1}{b-a}\ \psi(a) = 0$$

$$\Rightarrow\quad \psi(a) = \frac{(b-a)(b-\xi)^{n-1}}{(n-1)!}\ f^n(\xi) \ ....(2)$$

Putting $\xi = a + (b-a)\theta$ where $0 < \theta < 1$

We have $b-\xi = b-a-b\theta+a\theta = (1-\theta)(b-a)$

$\therefore (b-\xi)^{n-1} = (1-\theta)^{n-1}(b-a)^{n-1} = (1-\theta)^{n-1} h^{n-1}$ where $h = b-a$

$$\therefore \text{ from (2) we get } \psi(a) = \frac{h^n (1-\theta)^{n-1}}{(n-1)!}\ f^n(a+\theta h)$$

Replacing a by x, we get the expression for the remainder

$$R_n = \frac{h^n (1-\theta)^{n-1}}{(n-1)!}\ f^n(x+\theta h),\ 0 < \theta < 1$$

(ii) This is known as Cauchy's form of remainder in Taylor's expansion.

The Corresponding form the Maclaurin's expansion is

$$R_n = \frac{x^n (1-\theta)^{n-1}}{(n-1)!}\ f^n(\theta x) \quad 0 < \theta < 1$$

Remarks: Cauchy's form of remainder is sometimes more useful than that of Lagrange's form. The value of θ in the two forms of remainder for the same function need not be same.

8.14 Taylor's Infinite Series

If f(x), f′(x), $f^2(x)$,...... $f^n(x)$ exist finitely however large n may be in any interval [a, a+h] and if in additon R_n tends to zero as n tends to infinity i.e. $R_n \to 0$ as $n \to \infty$, then Taylor's series in infinite form is

$$f(a+h) = f(a) + h\,f'(a) + \frac{h^2}{2!}\ f^2(a) + ...\text{to } \infty$$

$$\text{Denoting } S_n = f(a) + h\,f'(a) + \frac{h^2}{2!}\ f^2(a) + ...\frac{h^{n-1}}{(n-1)!}\ f^{n-1}(a)$$

Then $f(x+h) = S_n + R_n$

Now, in addition $R_n \to 0$ as $n \to \infty$, then $\lim_{n\to\infty} S_n = f(x+h)$

$$\text{or } f(a) + h\, f'(a) + \frac{h^2}{2!} f^2(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) +$$

$$\ldots\ldots\text{to } \infty = f(a+h)$$

Corollary: Putting $h = x - a$ i.e $x = a + h$ we get another form of Taylor's Series which is

$$f(x) = f(a) + (x-a)\, f'(a) + \frac{(x-a)^2}{2!} f^2(a) + \ldots\ldots\text{to } \infty$$

8.15 Maclaurin's Infinite Series

When $a = 0$, $h = x$ in the theorem, we observe that if $f(x)$, $f'(x)$, $f^2(x)$,...$f^n(x)$ exists finintely however large n may be in any interval and if $R_n \to 0$ as $n \to \infty$, then Maclaurin's infinite series is

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{2!} f^2(x) + \ldots\ldots\text{to } \infty$$

Illustrative Examples

Example 1: If $f(h) = f(0) + h\, f'(0) + \frac{h^2}{2!} f^2(\theta h)$, $0 < \theta < 1$ find θ when $h=1$ and $f(x) = (1-x)^{5/2}$

Solution: We have $f(h) = (1-h)^{5/2}$ $\qquad \because f(x) = (1-x)^{5/2}$

$$\therefore \quad f'(h) = -\frac{5}{2}(1-h)^{3/2}$$

$$f^2(h) = \frac{15}{4}(1-h)^{1/2}$$

$$f(0) = 1,\ f'(0) = -\frac{5}{2}$$

$\therefore$ from the given relation, we get

$$(1-h)^{5/2} = 1 + h\left(-\frac{5}{2}\right) + \frac{h^2}{2!}\frac{15}{4}(1-\theta h)^{1/2}$$

$$\Rightarrow \quad 0 = 1 - \frac{5}{2} + \frac{15}{8}(1-\theta h)^{1/2} \text{ by putting } h = 1$$

$$\Rightarrow \quad (1-\theta h)^{1/2} = \frac{4}{5}$$

$$\Rightarrow \quad 1-\theta = \frac{16}{25} \Rightarrow \theta = \frac{9}{25}$$

Example 2: Prove that the Lagrange's remainder after n terms in the expansion of $e^{ax}\cos bx$ in powers of x is $\frac{(a^2+b^2)^{n/2}}{n!} x^n e^{a\theta x} \cos(b\theta x + n\tan^{-1}\frac{b}{a}), 0 < \theta < 1$

Solution: Lagrange's remainder after n terms in the expansion of f(x) is $\frac{x^n}{n!} f^n(\theta x)$ $0 < \theta < 1$

Here $f(x) = e^{ax}\cos bx$

$f^1(x) = ae^{ax}\cos bx - be^{ax}\sin bx$

$= e^{ax}(a\cos bx - b\sin bx)$

Putting $a = r\cos\phi$ and $b = r\sin\phi$ so that $r^2 = a^2 + b^2$ and $\tan\phi = \frac{b}{a}$ we get

$f'(x) = e^{ax}(r\cos\phi\cos bx - r\sin\phi\sin bx)$

$= re^{ax}\cos(bx + \phi)$

Similarly $f^2(x) = r^2 e^{ax}\cos(bx + 2\phi)$ and proceeding in this manner

$f^n(x) = r^n e^{ax}\cos(bx + n\phi)$

$= r^n e^{ax}\cos(bx + n\tan^{-1}\frac{b}{a})$

Hence Lagrange's Remainder after n terms is

$$\frac{x^n}{n!} f^n(\theta x) = r^n e^{a\theta x}\cos(b\theta x + n\tan^{-1}\frac{b}{a})\frac{x^n}{n!}$$

$$= \frac{\left(a^2+b^2\right)^{n/2}}{n!} x^n e^{a\theta x} \cos\left(b\theta x + n \tan^{-1}\frac{b}{a}\right) \; 0 < \theta < 1$$

$$\left[\because r = \left(a^2+b^2\right)^{1/2} \; r^n = \left(a^2+b^2\right)^{n/2}\right]$$

Example 3: Prove that Cauchy's remainder after n terms in the expansion of $(1+x)^m$ (m being a negative integer or fraction) in powers of x is

$$\frac{m(m-1)(m-2).....(m-n+1)}{(n-1)!} x^n (1+\theta x)^{m-1} \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \; 0 < \theta < 1$$

Solution: Cauchy's remainder after n terms in the expansion of f(x) is

$$\frac{x^n (1-\theta)^{n-1}}{(n-1)!} f^n(\theta), \; 0 < \theta < 1$$

Here $f(x) = (1+x)^m$

$\therefore \quad f^1(x) = m(1+x)^{m-1}$

$f^2(x) = m(m-1)(1+x)^{m-2}$

$\therefore \quad f^n(x) = m(m-1)(m-2).......(m-n+1)(1+x)^{m-n}$

Hence Cauchy's remainder after n terms in the expansion of $(1+x)^m$ is

$$\frac{m(m-1)(m-2).......(m-m+1)(1+\theta x)^{m-n}}{(n-1)!} x^n (1-\theta)^{n-1}$$

$$= \frac{m(m-1)(m-2).......(m-m+1)}{(n-1)!} x^n (1+\theta x)^{m-1} \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \; 0 < \theta < 1$$

Example 4: Prove that the Cauchy's remainder after n terms in the expansion of log (1+x) in powers of x is

$$(-1)^{n-1} \frac{x^n}{1+\theta x} \left(\frac{1-\theta}{1+\theta x}\right)^{n-1}, \; 0 < \theta < 1$$

Solution: Cauchy's remainder after n terms in the expansion of f(x) is

$$\frac{x^n (1-\theta)^{n-1}}{(n-1)!} f^n(\theta x), \; 0 < \theta < 1$$

Here f(x) = log (1+x)

$\therefore \quad f'(x) = \dfrac{1}{1+x} = (1+x)^{-1}$

$\therefore \quad f^2(x) = (-1)\,(1+x)^{-2} = (-1)^1.1\,(1+x)^{-2}$

$f^3(x) = (-1)\,(-2)\,(1+x)^{-3} = (-1)^2\,2!\,(1+x)^{-3}$

$\therefore \quad f^n(x) = (-1)^{n-1}\,(n-1)!\,(1+x)^{-n}$'

Hence Cauchy's remainder after n terms in the expansion of log(1+x) is

$$\frac{x^n(1-\theta)^{n-1}}{(n-1)!}\,(-1)^{n-1}\,(n-1)!\,(1+\theta x)^{-n}$$

$$= (-1)^{n-1}\,\frac{x^n}{1+\theta x}\,\frac{(1-\theta)^{n-1}}{1+\theta x},\ 0 < \theta < 1$$

Example 5: From the relation $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(\theta x)$, $0<\theta<1$ show that

$\log(1+x) > x - \frac{1}{2}x^2$, if $x>0$ (NEHU 2014)

Solution: Let f(x) = log (1+x). Then

$f'(x) = \dfrac{1}{1+x} = (1+x)^{-1}$

$f''(x) = f^2(x) = (-1)\,(1+x)^{-2} = (-1)^{2-1}\,(1+x)^{-2}$

$f'''(x) = f^3(x) = (-1)\,(-2)\,(1+x)^{-3} = (-1)^{3-1}\,2!\,(1+x)^{-3}$

$f^4(x) = (-1)\,(-2)\,(-3)\,(1+x)^{-4} = (-1)^{4-1}\,3!\,(1+x)^{-4}$

$\therefore \quad f^n(x) = (-1)^{n-1}\,(n-1)!\,(1+x)^{-n} = \dfrac{(-1)^{n-1}(n-1)!}{(1+x)^n}$

$\therefore$ From the relation $f(x) = f(0) + x\,f'(0) + \dfrac{x^2}{2!}\,f''(0)$

We get $\log(1+x) = \log 1 + x.1 + \dfrac{x^2}{2!}(-1)$

$\Rightarrow \quad \log(1+x) > x - \dfrac{x^2}{2!}$

Example 6: Expand sinx in a finite series in powers of x with remainder in Lagrange's form (NEHU 2003, 2007, 2016)

Solution: Let f(x) = sinx then

$$f'(x) = \cos x = \sin\left(\tfrac{\pi}{2}+x\right)$$

$$f^2(x) = -\sin x = \sin\left(2\tfrac{\pi}{2}+x\right)$$

$$f^3(x) = -\cos x = \sin\left(3\tfrac{\pi}{2}+x\right)$$

$$\therefore \quad f^n(x) = \sin\left(n\tfrac{\pi}{2}+x\right)$$

Also $f^n(0) = \sin\frac{n\pi}{2}$ which is 0 or ±1 according as n is even or odd

$$\text{Now } R_n = \frac{x^n}{n!} f^n(\theta x) = \frac{x^n}{n!}\sin\left(\frac{n\pi}{2}+\theta x\right)$$

$$|R_n| = \left|\frac{x^n}{n!}\right| \left|\sin\left(\frac{n\pi}{2}+\theta x\right)\right| \le \left|\frac{x^n}{n!}\right| \text{ as } \left|\sin\frac{n\pi}{2}+\theta x\right| \le 1$$

$$\therefore \quad R_n \to 0 \text{ as } n \to \infty, \text{ since } \frac{x^n}{n!} \to 0 \text{ as } n \to \infty$$

∴ By Maclaunn's series in finite forms we get

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{2!} f^2(0) + \ldots\ldots + \frac{x^n}{n!} f^n(\theta x) \quad 0 < \theta < 1$$

$$\text{i.e } \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots.. \frac{x^n}{n!} \sin\left(\frac{n\pi}{2}+\theta x\right) \quad 0 < \theta < 1$$

Which is expansion of sinx in powers of x with Lagrang's remainder

As $R_n \to 0$ as $n \to \infty$ the condition for Maclaumn's expansion are also satisfies

$$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots.. \text{ to } \infty \text{ for all values of x.}$$

Example 7: Expand cosx in a finite series in powers of x with remainder in Lagrange's form (NEHU 2008)

Solution: Let f(x) = cosx. Then

$$f'(x) = \cos x = \sin (\pi/2+x)$$

$$f^2(x) = -\cos x = \cos (2\pi/2+x)$$

$$f^3(x) = \sin x = \cos (3\pi/2+x)$$

$$\therefore\ f^n(x) = \cos (n\pi/2+x)$$

By Maclaurin's series in finite form we get

$$f(x) = f(0) + x\, f^1(0) + \frac{x^2}{2!}\, f^2(0)+......+\frac{x^n}{n!}\, f^n(\theta x)\ 0 < \theta < 1$$

i.e $$\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}\ .....\ +\frac{x^n}{n!}\cos\left(\frac{n\pi}{2}+\theta x\right)\ 0 < \theta < 1$$

Where the Lagrange's remainder is given by

$$R_n = \frac{x^n}{n!}\, f^n\,(\theta x) = \frac{x^n}{n!}\cos\left(\frac{n\pi}{2}+\theta x\right)$$

As $R_n \to 0$ as $n \to \infty$ since $\frac{x^n}{n!} \to 0$ as $\to \infty$

The condition for Maclaumn's infinite are satisfied

$$\therefore\ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}\ .....\ \text{to } \infty \text{ for all values of x.}$$

Example 8: Expand $(1+x)^m$ in a finite series in power of x with remainder in Lagrange's from (NEHU 2004)

Solution: Let $f(x) = (1+x)^m$ Then

$$f'(x) = m\,(1+x)^{m-1}$$

$$f^2(x) = m(m-1)\,(1+x)^{m-2}$$

$$f^3(x) = m(m-1)\,(m-2)\,(1+x)^{m-3}$$

$$\therefore\ f^n(x) = m(m-1)\,(m-2).....(m-n+1)\,(1+x)^{m-n}$$

By Maclaurin's series in finite forms with Cauchy's form of remainder we get

$$f(x) = f(0) + x\, f^1(0) + \frac{x^2}{2!}\, f^2(0)+......+\frac{x^n(1-\theta)^{n-1}}{(n-1)!}\, f^n(\theta x)$$

$$= 1 + mx + \frac{x^2}{2!}\, m(m-1) + \frac{x^3}{3!}\, m(m-1)\,(m-2) + \ldots\ldots$$

$$+ \frac{x^n}{(n-1)!}\,(1-\theta)^{n-1}.m(m-1)\;....\;(m-n+1)\,(1+\theta x)^{m-n}$$

Where Cauchy remainder is given by

$$R_n = \frac{x^n}{(n-1)!}\,(1-\theta)^{n-1}\, f^n(\theta x)$$

$$= m(m-1).....(m-n+1)\,\frac{x^n}{(n-1)!}\,(1-\theta)^{n-1}\,(1+\theta x)^{m-n},\; 0<\theta<1$$

Example 9: Expand log(1+x) in a finite series in powers of x with Cauchy's form of remainder (NEHU 2016)

Solution: Let f(x) = log (1+x)

Same as example 4.

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3}\;\ldots\ldots + (-1)^{n-1}\frac{x^n}{1+\theta x}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}\quad 0<\theta<1$$

Exercises

1. Find the value of ξ in the Mean Value Theorem

   $f(b) - f(a) = (b-a)\, f'(\xi)$

   (i) If $f(x) = x^2$, a = 1, b = 2

   (ii) If $f(x) = \sqrt{x}$, a = 4, b = 9

   (iii) If $f(x) = x(x-1)(x-2)$; a = 0, b = $\frac{1}{2}$

   (iv) If $f(x) = Ax^2 + Bx + C$ in (a, b)

2. In the Mean Value Theorem

   $f(x+h) = f(x) + hf'(x+\theta h)$

   If $f(x) = Ax^2 + Bx + C$, where $A \neq 0$. Show that $\theta = \frac{1}{2}$

3. In the Mean Value Theorem

$f(a+h) = f(a) + hf'(x+\theta h)$

If $a = 1$, $h = 3$ and $f(x) = \sqrt{x}$, find θ

4. If $f(h) = f(0) + hf'(0) + \frac{h^2}{2!} f^2(\theta h)$ $0 < \theta < 1$ find θ when $h = 7$ and $f(x) = \frac{1}{1+x}$

5. From the relation

$$f(h) = f(0) + xf'(0) + \frac{x^2}{2!} f^2(\theta x) \quad 0 < \theta < 1$$

Show that $\log(1+x) > x - \frac{1}{2}x^2$ if $x>0$

and $\cos x > 1-\frac{1}{2}x^2$ if $0 < x < \frac{1}{2}\pi$

6. If $f(x) = \tan x$, then $f(0) = 0 = f(\pi)$. If Rolle's theorem applicable to $f(x)$ in $(0, \pi)$?

7. Show that

$$(x+h)^{3/2} = x^{3/2} + \frac{3}{2}x^{1/2}h + \frac{3.1}{2.2} \cdot \frac{h^2}{2!} \frac{1}{\sqrt{x+\theta h}} \quad 0 < \theta < 1$$

Find θ when $x=0$

8. Expand in a finite series in powers of h and find the remainder in each case.

(i) $\log(x+h)$ (ii) $\sin(x+h)$ (iii) $(x+h)^m$

9. Apply Taylor's theorem to obtain the Binomial expansion of $(a+h)^n$, where n is a positive integer.

10. If f(x) is a polynomial of degre r, then show that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f^2(a) + \ldots\ldots + \frac{h^r}{r!} f^r(a)$$

11. Expand $5x^2 + 7x + 3$ in powers of $(x-2)$

12. Expand the following functions in a finite series in powers of n with remainder in Lagrange's form in each case

(i) e^x (ii) a^x (iii) $\sin x$ (iv) $\cos x$ (v) $\log(1+x)$

(vi) $\log(1-x)$ (vii) $(1+x)^m$ (vii) $e^x \cos x$ (ix) $e^{ax} \sin bx$

13. Expand the following functions in finite series in powers of x with remainder in Cauchy's form in each case:

    (i) e^x (ii) cosx (iii) $\frac{1}{1-x}$

14. In Cauchy's Mean Value Theorem; show that θ is independent of both x and h and equal to $\frac{1}{2}$ in each of tghe following case:

    (i) If $\phi(x) = e^x$ and $\psi(x) = e^{-x}$

    or (ii) If $\phi(x) = \sin x$ and $\psi(x) = \cos^x$

    or (iii) If $\phi(x) = x^2 + x + 1$ and $\psi(x) = 2x^2 + 3x + 4$

15. If $f(x) = x^2$, $\phi(x) = x$, then find the value of ξ in terms of a and b in Cauchy's Mean Value Theorem.

16. If f(x) and g(x) are differentiable in the interval (a, b), then prove that there is a number ξ, $a<\xi<b$ such that

$$\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a) \begin{vmatrix} f(a) & f'(\xi) \\ g(a) & g'(\xi) \end{vmatrix}$$

17. Expand in infinite series in powers of h:

    (i) e^{x+h} (ii) cos(x+h) (iii) sin(x+h) (iv) log(x+h)

18. Show that $\sqrt{x}$ and $x^{5/2}$ cannot be expanded in Madaurin's infinite series.

19. Show that $f(x) = x^{3/2}$ cannot be expanded in Madaurin's infinite series or that for this function the expansion f(x+h) fails when x=0, but that there exists a proper fraction θ such that

$$f(x+h) = f(x) + hf^1(x) + \frac{1}{2} h^2 f^2(x+\theta h) \text{ holds when } x=0$$

20. Find the conditions under which a function can be expanded in Taylor's infinite series

9

Maxima and Minima

Introduction

Maxima and Minima is one of the most important branch of calculus and has various practical applications to Geometry, Physics and other sciences.

9.1 Definition

A function f(x) is said to have a maximum (or local maximum) at the point x=a, if $f(a) \geq f(x)$ for all values of x in some suitably small neighbourhood of a i.e $f(a+h) \leq f(a)$ for $|h|$ sufficiently small.

Similarly, a function f(x) is said to have a minimum (or local minimum) at x=a if $f(a) \leq f(x)$ for all values of x in some suitably small neighnourhood of a

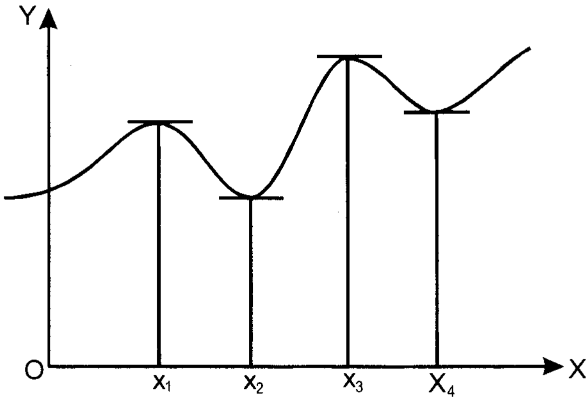
i.e $f(a+h) \geq f(a)$, for $|h|$ sufficiently small.

The figure below represents graphically the continuous function f(x) with maximum values at x_1, x_3 and minimum values at x_2, x_4 observe that, maximum value of f(x) at x_1 is less than minimum value of f(x) at x_4. The greatest value of f(x) is assumed at x_3.

They are respectively the absolute maximum and the absolute minimum of f(x).

From the figure the following features regarding maxima and minima of a continuous function are apparent.

(i) that the function may have several maxima and minima



(ii) that the maximum value of the function at some point may be less than the minimum value of it at another point.

(iii) the maximum and minimum values of the functioin occur alternately i.e between any two consecutive maximum values there is a minimum value and vice versa.

9.2 A Necessary Condition for Maximum and Minimum

Theorem: If f(x) be a maximum, or a minimum at x=c and if f′(c) exists, then $f^1(c) = 0$

Proof: By definition, f(x) is maximum at x=c provided we can find a positive number δ such that

$f(c+h) - f(c) < 0$ when evern $-\delta<h<\delta$ $(h\neq 0)$

$$\therefore \frac{f(c+h)-f(c)}{h} < 0 \text{ if h is positive and sufficiently small and}$$

$$\frac{f(c+h)-f(c)}{h} > 0 \text{ if h is negative and numerically sufficiently small.}$$

$$\text{Hence, } \lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h} \leq 0 \text{ and } \lim_{h\to 0^-}\frac{f(c+h)-f(c)}{h} \geq 0$$

Now if f′(c) exists, the above two limits must be equal.

Hence the only posiibility is that f′(c) = 0

9.3 Observations

(i) f′(c) = 0, if it exists is a necessary but not a sufficient condition for f(x) to have extreme value at x=c

For example, if $f(x) = x^3$, then at x=0 $f'(0) = 0$

But $f(x) > f(o)$ when $x>0$ and $f(x) < f(0)$ when $x<0$

Hence f(x) has no extreme value at x=0

(ii) Even when $f'(c)$ does not exists, f(c) may be a maximum or a minimum

For example, if $f(x) = |x|$, then clearly f(0) is a minimum value but $f'(0)$ does not exists.

(iii) The extremes for which $f'(x) = 0$ correspond to a point where the tangent to the curve y=f(x) is parallel to x=axis

(iv) The point where $f'(x) = 0$ is generally called a stationary point.

9.4 Conditions for Maxima and Minima

Let f(x) be a function which can be expanded in the neighbourhood of x=a, by Taylor's Theorem

At x=a, the value of f(x) is f(a)

Let us consider two values of x, viz a+h and a–h in the neighbourhood and on either side of x=a, h being very small.

If there is a minimum at x=a, then from definition $f(a) < f(a+h)$ and $f(a) < f(a-h)$

i.e $f(a+h) - f(a)$ and $f(a-h) - f(a)$ are both positive for minumum at x=a

If there is a maximum at x=a, then from definition $f(a) > f(a+h)$.

i.e $f(a+h) - f(a)$ and $f(a-h) - f(a)$ are both negative for maximum at x=a

Now by Taylor's Theorem

$$f(a+h) = f(a) + h\, f'(a) + \frac{h^2}{2!} f^2(a) + \frac{h^3}{3!} f^3(a) + \ldots\ldots.$$

$$\text{or} \quad f(a+h) - f(a) = h\, f'(a) + \frac{h^2}{2!} f^2(a) + \frac{h^3}{3!} f^3(a) + \ldots\ldots. \quad \text{(i)}$$

$$\text{Similarly,} \quad f(a-h) = f(a) - h\, f'(a) + \frac{h^2}{2!} f^2(a) - \frac{h^3}{3!} f^3(a) + \ldots\ldots.$$

$$\text{or} \quad f(a-h) - f(a) = -h\, f'(a) + \frac{h^2}{2!} f^2(a) + \frac{h^3}{3!} f^3(a) + \ldots\ldots. \quad \text{(ii)}$$

Since h is very small we may neglet power of h higher than the first and so from (i) and (ii) we get

$$f(a+h) - f(a) = h\ f'(a) \text{ ....(iii)}$$

and $$f(a-h) - f(a) = -h\ f'(a) \text{ ....(iv)}$$

We have seen that for maximum or minimum, the sign of f(a+h) – f(a) and f(a–h) – f(a) must be the same.

So from (iii) and (iv) we conclude that f(a+h) – f(a) and f(a–h) – f(a) have the same sign if f'(a) is zero (otherwise they will have different signs)

Hence the necessary condition that f(x) should have a maximum or a minumum at x=a is f'(a) = 0

Now if f'(a) = 0 we have from (i) and (ii)

$$f(a+h) - f(a) = \frac{h^2}{2!} f^2(a) \text{ ......(v)}$$

and $$f(a-h) - f(a) = \frac{h^2}{2!} f^2(a) \text{ ......(vi)}$$

neglectinig powers of h higher than the second since h is very small, we find that f(a+h) – f(a) and f(a–h) – f(a) are both of the same sign.

Now two cases arise:

(i) When $f^2(a)$ is positive. In this case both f(a+h) – f(a) and f(a–h) – f(a) are positive and hence there is a minimum at x=a

(ii) When $f^2(a)$ is negative. In this case both f(a+h) – f(a) and f(a–h) – f(a) are negative and hence there is a maximum at x=a

Hence we conclude that:

(i) The function f(x) is maximum at x=a if f'(a)=0 and $f^2(a)=f''(a)<0$ and

(ii) The function f(x) is minimum at x=a if f'(a)=0 and $f^2(a) = f''(a)>0$

9.5 Application to Problems

In solving a problem in which the maximum or a minimum value of a quantity is needed, we should always be guided by the following important principle.

Express the quantity of which the maximum of minumum value is required, as a function of one unknown.

In some cases, it may be necessary to express the quantity in terms of two or more unknowns and then by means of a given condition to express all these unknowns interms of one of them.

In connection with problems concerning spheres, cones and cylinder the following results would be needed

1. Sphere of radius r

   Volume = $\frac{4}{3}\pi r^3$. Surface Area = $4\pi r^2$

2. Circular Cylinder of height h and radius of base r

   Volume = $\pi r^2 h$. Curved Surface Area = $2\pi rh$

   Area of each plane face = πr^2

3. Right circular cone of height h and radius of base r

   Semi vertical angle = $\tan^{-1}\left(\frac{r}{h}\right)$

   Slant height = $\sqrt{r^2+h^2}$

   Volume = $\frac{1}{3}\pi r^2 h$. Curved Surface Area = $\pi r\sqrt{r^2+h^2}$

Illustrative Examples

Example 1. For what values of x, the following expression is maximum or minimum respectively: $2x^3 - 21x^3 + 36x - 20$

Solution: Let $f(x) = 2x^3 - 21x^3 + 36x - 20$

$\therefore f^1(x) = 6x^2 - 42x + 36$, which exists for all values of x

Now for maximum or minimum, $f'(x) = 0$

i.e $6x^2 - 42x + 36 = 0$

$\Rightarrow x^2 - 7x + 6 = 0$

$\Rightarrow (x - 1)(x - 6) = 0$

$\Rightarrow x = 1$ or 6

Again $f^2(x) = f''(x) = 12x - 42 = 6(2x-7)$

Now when x=1, $f''(x) = -30 < 0$

When x=6, $f''(x) = 30>0$

Hence f(x) is maximum at x=1 and minimum at x=6

The maximum value of $f(x) = 2.1^3 - 21.1^2 + 36.1 - 20 = -3$

The minimum value of $f(x) = 2.6^3 - 21.6^2 + 36.6 - 20 = -128$

Example 2: Show that $x^3 - 3x^2 + 3x + 7$ has neither a maximum nor a minimum at x=1

Solution: Let $f(x) = x^3 - 3x^2 + 3x + 7$

$\therefore \quad f'(x) = 3x^2 - 6x + 3$

$f^2(x) = f''(x) = 6x - 6$

For maximum or minimum, $f^1(x) = 0$

i.e $\quad 3x^2 - 6x + 3 = 0$

$\Rightarrow \quad x^2 - 2x + 1 = 0$

$\Rightarrow \quad (x - 1)^2 = 0$

$\Rightarrow \quad x = 1$

Now at x=1, $f''(x) = 6-6=0$

Hence the given function has neither a maximum nor a minimum at x=1

Example 3: Show that the maximum value of $x + \frac{1}{x}$ is less than the minimum value

Solution: Let $f(x) = x + \frac{1}{x}$

$$\therefore \quad f'(x) = 1 - \frac{1}{x^2}$$

$$f^2(x) = f''(x) = = 0 + \frac{2}{x^3} = \frac{2}{x^3}$$

Now for maxima or minima, $f'(x) = 0$

$$\Rightarrow 1 - \frac{1}{x^2} = 0 \quad \Rightarrow x = \pm 1$$

When $x = -1$, $f''(x) = -2 < 0$

When $x = 1$, $f''(x) = 2 > 0$

Hence f(x) is maximum at x=–1 amd minimum at x=1

The maximum value of $f(x) = -1 + \frac{1}{-1} = -2$

The minimum value of $f(x) = 1 + \frac{1}{1} = 2$

Hence the maxiumum value of f(x) is less than the minimum value

Example 4: Show that x^x is minimum for $x = \frac{1}{e}$ (NEHU 2003)

Solution: Let $f(x) = x^x$

$\Rightarrow \quad \log f(x) = \log x^x$

$\Rightarrow \quad \log f(x) = x\log x$

Differentiating both sides w.r.t x

$$\frac{d}{dx}\{\log f(x)\} = \frac{d}{dx}\{x\log x\}$$

$$\Rightarrow \quad \frac{1}{f(x)}\frac{d}{dx}f(x) = 1+\log x$$

$$\Rightarrow \quad f'(x) = f(x)\,(1+\log x)$$

Again $\frac{d}{dx}f'(x) = \frac{d}{dx}\{f(x)\,(1+\log x)\}$

$$\Rightarrow \quad f''(x) = f'(x)\,(1+\log x) + f(x)\,\frac{1}{x}$$

$$\Rightarrow \quad f''(x) = f(x)\,(1+\log x)^2 + f(x).\frac{1}{x}$$

$$\Rightarrow \quad f''(x) = f(x)\,[(1+\log x)^2 + \frac{1}{x}]$$

$$\Rightarrow \quad f''(x) = x^x\,[(1+\log x)^2 + \frac{1}{x}]$$

For maximum or minima $f'(x) = 0$

$\Rightarrow \quad f(x)\,(1+\log x) = 0$

$\Rightarrow \quad 1 + \log x = 0 \qquad \because f(x) \neq 0$

$\Rightarrow \quad \log x = -1$

$\Rightarrow \quad x = e^{-1} = \frac{1}{e}$

Putting $x = \frac{1}{e}$ in $f''(x)$ we get

$$f''(x) = \left(\frac{1}{e}\right)^{1/e}.e > 0$$

Hence x^x is minimum for $x = \frac{1}{e}$

Example 5. Divide a number 15 into two parts such that the square of one multiplied with the cube of the other is maximum.

Solution: Let x be one of the parts. Then 15–x is the other part

Let $P = x^2 (15-x)^3$

Then $\frac{dP}{dx} = 2x (15-x)^3 - 3x^2 (15-x)^2$

and $\frac{d^2P}{dx^2} = 2(15-x)^3 - 6x (15-x)^2 - 6x (15-x)^2 + 6x^2 (15-x)$

$\Rightarrow \quad \frac{d^2P}{dx^2} = 2(15-x) [(15-x)^2 - 6x (15-x) + 3x^2]$

Now for maxima or minima $\frac{dP}{dx} = 0$

i.e $\quad 2x (15-x)^3 - 3x^2 (15-x)^2 = 0$

$\Rightarrow \quad x (15-x)^2 [30-5x] = 0$

$\Rightarrow \quad x = 0, (15-x)^2 = 0, 30-5x = 0$

$\Rightarrow \quad x = 0, 15, 6$

$\Rightarrow \quad x = 6 \qquad (\because x = 0, 15 \text{ is not possible})$

When x=6, $\frac{d^2P}{dx^2} = 2(15-6) [(15-6)^2 - 6.6 (15-6) + 3.6^2]$

$= 2.9 [9^2 - 324 + 108] < 0$

Hence for P to be maximum, x=6

$\therefore$ The two parts are 6 and 9.

Example 6. Show that the maximum rectangle with a given permeter is a square.
(NEHU 2016)

Solution: Let x and y be the length and breadth of the rectangle. Then

Perimeter $P = 2(x+y)$ ....(i)

and Area $A = xy$ .... (ii)

Hence Area $A = x \left[\frac{P}{2} - x\right]$

$$\therefore \frac{dA}{dx} = \left[\frac{P}{2} - x\right] - x = \frac{P}{2} - 2x$$

$$\therefore \frac{d^2A}{dx^2} = -2$$

Now for maxima or minima, $\frac{dA}{dx} = 0$

$$\text{i.e } \frac{P}{2} - 2x = 0$$

$$\Rightarrow \frac{P}{2} = 2x \Rightarrow x = \frac{P}{4}$$

Putting $x = \frac{P}{4}$ in $\frac{d^2A}{dx^2}$ we get

$$\frac{d^2A}{dx^2} = -2 < 0$$

Also when $x = \frac{P}{4}$, from (i) we get

$$P = 2\left(\frac{P}{4} + y\right)$$

$$\Rightarrow \frac{P}{2} = \frac{P}{4} + y$$

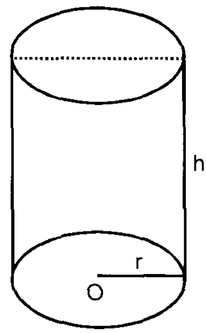
$$\Rightarrow y = \frac{P}{2} - \frac{P}{4} = \frac{P}{4}$$

$$\therefore x = y = \frac{P}{4}$$

Hence the rectangle is maximum, if length = breadth i.e it is a square

Example 7. Show that the right circular cylinder of given surface is of maximum volume when to height is equal to the diameter of the base. (NEHU 2013)

Solution: Let h be the height and r the radius of the base of a right circular cylinder.



Then surface area $S = 2\pi r^2 + 2\pi rh$ ....(i)

and volume $V = \pi r^2 h$ ....(ii)

By (i) and (ii) we get

$$V = \pi r^2 \left(\frac{S - 2\pi r^2}{2\pi r}\right) = \frac{Sr - 2\pi r^3}{2}$$

$$\therefore \quad \frac{dV}{dr} = \frac{1}{2}(S - 6\pi r^2)$$

$$\Rightarrow \quad \frac{d^2V}{dr^2} = \frac{1}{2}(-12\pi r) = -6\pi r \qquad [\because S \text{ is a constant}]$$

For maxima or minima, $\frac{dV}{dr} = 0$

i.e $\frac{1}{2}(S - 6\pi r^2) = 0$

$$\Rightarrow \quad S = 6\pi r^2 \Rightarrow r = \sqrt{\frac{S}{6\pi}} \qquad [\because r \text{ being positive}]$$

When $r = \sqrt{\frac{S}{6\pi}}$, we get $\frac{d^2V}{dr^2} = -3\pi\frac{S}{6\pi} = -\frac{S}{2} < 0$

Hence volume is maximum when $r = \sqrt{\frac{S}{6\pi}}$

Now from (i) $S = 2\pi r^2 + 2\pi rh$

$\Rightarrow \pi r^2 = 2\pi rh$

$\Rightarrow r = 2h$

Thus volume of a right circular cylinder of given surface is maximum when its height is equal to the diameter of the base

Example 8. Show that of all rectangles of given area, the square has the least perimeter. (NEHU 2016)

Solution: Let x and y be the length and breadth of a rectangle then

Area $A = xy$ ....(i)

Perimeter $P = 2(x+y)$ ....(ii)

By (i) and (ii) $P = 2\left(x + \frac{A}{x}\right)$ $\because$ A is constant

$$\therefore \quad \frac{dP}{dx} = 2\left(1 - \frac{A}{x^2}\right)$$

$$\Rightarrow \quad \frac{d^2P}{dx^2} = \frac{4A}{x^3}$$

For maxima or minima, $\frac{dP}{dx} = 0$

i.e $\quad 2\left(1 - \frac{A}{x^2}\right) = 0$

$$\Rightarrow \quad \left(1 - \frac{A}{x^2}\right) = 0 \Rightarrow \frac{A}{x^2} = 1 \Rightarrow x = \pm\sqrt{A}$$

$\therefore x = \sqrt{A}$ (Area being positive)

When $x = \sqrt{A}$, $\frac{d^2P}{dx^2} = \frac{4A}{A\sqrt{A}} > 0$

Hence Penmeter is minimum when $x = \sqrt{A}$

Now from (i) $A = \sqrt{A}\ y \Rightarrow y = \sqrt{A}$

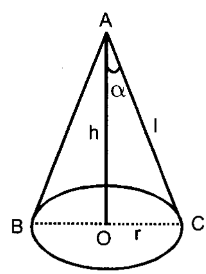
$\therefore x = y = \sqrt{A}$

Hence of all rectangles of given area, the square has the least perimeter

Example 9. Show that the semi vertical angle of the cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$

Solution: Let h, r, l and α be the height, radius of the base, slant height and semi vertical angle of the cone. Then from figure $r = l\sin\alpha$ and $h = l\cos\alpha$

Volume of the cone $V = \frac{1}{3}\pi r^2 h$



$$\Rightarrow V = \frac{1}{3}\pi l^3 \sin^2\alpha \cos\alpha$$

$$\therefore \quad \frac{dV}{d\alpha} = \frac{1}{3}\pi l^3[2\sin\alpha \cos^2\alpha - \sin^2\alpha]$$

$$= \frac{1}{3}\pi l^3 \sin\alpha\ [2\cos^2\alpha - \sin^3\alpha]$$

and $\dfrac{d^2V}{d\alpha^2} = \dfrac{1}{3}\pi l^3 \cos\alpha\ [2\cos^2\alpha - \sin^2\alpha]$

$$+ \frac{1}{3}\pi l^3 \sin\alpha\ [-4\cos\alpha\sin\alpha - 2\sin\alpha\cos\alpha]$$

$\Rightarrow \quad \dfrac{d^2V}{d\alpha^2} = \dfrac{1}{3}\pi l^3\ [2\cos^3\alpha - \cos\alpha\sin^2\alpha - 6\sin^2\alpha\cos\alpha]$

$$= \frac{1}{3}\pi l^3 [2\cos^3\alpha - 7\cos\alpha\sin^2\alpha]$$

For maximum or minimum, $\dfrac{dV}{d\alpha} = 0$

i.e $\quad \dfrac{1}{3}\pi l^3 \sin\alpha\ [2\cos^2\alpha - \sin^2\alpha] = 0$

$\Rightarrow \quad \sin\alpha = 0$ or $2\cos^2\alpha = \sin^2\alpha$

$\Rightarrow \quad \alpha = 0$ or $\tan^2\alpha = 2 \Rightarrow \alpha = \tan^{-1}\sqrt{2}$

$\therefore \quad \alpha = \tan^{-1}\sqrt{2} \qquad (\because \alpha = 0$ is absurd)

When $\alpha = \tan^{-1}\sqrt{2}$

$$\frac{d^2V}{d\alpha^2} = \frac{1}{3}\pi l^3 [2\cos^2\alpha.\cos\alpha - 7\cos\alpha.\sin^2\alpha]$$

$$= \frac{1}{3}\pi l^3 [\sin^2\alpha.\cos\alpha - 7\cos\alpha.\sin^2\alpha]$$

$$= \frac{1}{3}\pi l^3 [-6\cos\alpha\sin^2\alpha]$$

$= -2\pi l^3 \cos\alpha\sin^2\alpha \qquad \because \tan^2\alpha = 2 \Rightarrow 1+\tan^2\alpha = 3$

$= -2\pi l^3 \dfrac{1}{\sqrt{3}}.\dfrac{2}{3} \qquad \Rightarrow \sec^2\alpha = 3 \Rightarrow \cos\alpha = \dfrac{1}{\sqrt{3}}$

$= -\dfrac{4\pi l^3}{3\sqrt{3}} < 0 \qquad \Rightarrow \sin^2\alpha = 1 - \dfrac{1}{\sqrt{3}} = \dfrac{2}{3}$

$\therefore$ Volume of the cone is maximum when $\alpha = \tan^{-1}\sqrt{2}$

Example 10. Show that the right circular cylinder of given volume which is open at the top has minimum total surface are, provided its height is equal to the radius of the base. (NEHU 2015)

Solution: Let he be the height and r the radius of the base of the cylinder. Then

Volume $V = \pi r^2 h$ ....(i)

and Surface Area $S = \pi r^2 + 2\pi rh$ ....(ii) ($\because$ open at the top)

$\therefore$ By (i) and (ii) we get

$$S = \pi r^2 + \frac{2V}{r}$$

$$\therefore \quad \frac{ds}{dr} = 2\pi r - \frac{2V}{r^2}$$

and $$\frac{d^2s}{dr^2} = 2\pi + \frac{4V}{r^3} \qquad (\because V \text{ is constant})$$

For maxima or minima, $\frac{ds}{dr} = 0$

$$\text{i.e } 2\pi r - \frac{2V}{r^2} = 0$$

$$\Rightarrow \quad \pi r = \frac{V}{r^2} \Rightarrow V = \pi r^3$$

$\therefore$ When $V = \pi r^3$, $\frac{d^2s}{dr^2} = 2\pi + \frac{4\pi r^3}{r^3} = 6\pi > 0$

$\therefore$ Surface area of a cylinder is minimum when $V = \pi r^3$

Now from (i) $\pi r^3 = \pi r^2 h \Rightarrow r = h$

Hence surface area of the cylinder is minimum if its height is equal to radium of the base

Example 11. Show that $x^2 \log\left(\frac{1}{x}\right)$ has a local maximum at $x = \frac{1}{\sqrt{e}}$ and its maximum value is $\frac{1}{2e}$ (NEHU 2004)

Solution: Let $f(x) = x^2 \log\left(\frac{1}{x}\right)$

$$\therefore \quad f'(x) = \frac{d}{dx}[x^2 \log\frac{1}{x}]$$

$$= 2x \log\frac{1}{x} + x^2.x\left(-\frac{1}{x^2}\right)$$

$$= 2x \log\frac{1}{x} - x$$

$$f^2(x) = f''(x) = \frac{d}{dx}[2x \log\frac{1}{x} - x]$$

$$= 2 \log\frac{1}{x} + 2x.x\left(-\frac{1}{x^2}\right) - 1$$

$$= 2x \log\frac{1}{x} - 3$$

For maxima or minima, $f'(x) = 0$

$$\text{i.e } 2x \log\frac{1}{x} - x = 0$$

$$\Rightarrow 2 \log\frac{1}{x} = 1 \Rightarrow \log\frac{1}{x} = \frac{1}{2}$$

$$\Rightarrow \frac{1}{x} = e^{1/2} = \sqrt{e}$$

$$\therefore x = \frac{1}{\sqrt{e}}$$

When $x = \frac{1}{\sqrt{e}}$, $f^2(x) = f''(x) = -1 < 0$

Hence f(x) is maximum when $x = \frac{1}{\sqrt{e}}$

Maximum value of $f(x) = \left(\frac{1}{\sqrt{e}}\right)^2 \left(\log\sqrt{e}\right) = \frac{1}{e}.\frac{1}{2} = \frac{1}{2e}$

Example 12. Show that f(x) = 12 (logx + 1) + x^2 – 10x + 3 is maximum when x=2 and minimum when x=3. (NEHU 2008)

Solution: f(x) = 12 (logx + 1) + x^2 – 10x + 3

$$\therefore \quad f'(x) = 12.\frac{1}{x} + 2x - 10$$

$$f^2(x) = f''(x) = -\frac{12}{x^2} + 2$$

For maximum or minimum, f′(x) = 0

$$\text{i.e} \quad \frac{12}{x} + 2x - 10 = 0$$

$$\Rightarrow \quad 2x^2 - 10x + 12 = 0$$

$$\Rightarrow \quad (x-2)(x-3) = 0$$

$$\therefore x = 2, 3$$

When x=2, $f''(x) = 2 - \frac{12}{4} = -1 < 0$

When x=3, $f''(x) = 2 - \frac{12}{9} = \frac{2}{3} > 0$

Hence f(x) is maximum when x=2 and minimum when x=3

Example 13. Show the maximum value of $\left(\frac{1}{x}\right)^x$ is $e^{1/e}$ (NEHU 2007)

Solution: Let $f(x) = \left(\frac{1}{x}\right)^x$

$$\therefore \log f(x) = x \log\left(\frac{1}{x}\right)$$

Differentiating both sides w.r.t x

$$\frac{1}{f(x)}\frac{d}{dx}f(x) = \frac{d}{dx}(x \log\frac{1}{x})$$

$$\Rightarrow \quad \frac{1}{f(x)}f'(x) = \log\frac{1}{x} + x.x\left(\frac{-1}{x^2}\right) = \log\frac{1}{x} - 1$$

$$\Rightarrow \quad f'(x) = f(x)\,[\log \frac{1}{x} - 1]$$

$$\therefore \quad f''(x) = \frac{d}{dx}\,\{f(x)\,[\log \frac{1}{x} - 1]\}$$

$$\Rightarrow \quad f''(x) = f'(x)\,[\log\frac{1}{x} - 1] - f(x).x\left(-\frac{1}{x^2}\right)$$

$$\Rightarrow \quad f''(x) = f(x)\,[\log\frac{1}{x} - 1]^2 - f(x)\frac{1}{x} = f(x)\,[(\log\frac{1}{x} - 1)^2 - \frac{1}{x}]$$

$$= \left(\frac{1}{x}\right)^x [(\log\frac{1}{x} - 1)^2 - \frac{1}{x}]$$

For maximum or minimum, $f'(x) = 0$

i.e $\quad f(x)\,[\log\frac{1}{x} - 1] = 0$

$$\Rightarrow \quad \log\frac{1}{x} - 1 = 0 \qquad \because f(x) \neq 0$$

$$\Rightarrow \quad \log\frac{1}{x} = 1 \Rightarrow \frac{1}{x} = e \Rightarrow x = \frac{1}{e}$$

When $x = \frac{1}{e}$, $f''(x) = (e)^{1/e}\,[0-e] = -\,e\left(e^{1/e}\right) < 0$

Hence f(x) is maximum when $x = \frac{1}{e}$

Maximum value of $f(x) = (e)^{1/e}$

Example 14. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Find the volume of the largest cylinder inscribed in a sphere of radius R.

Solution: Let r be the radius of the base and h the height of the cylinder inscribed

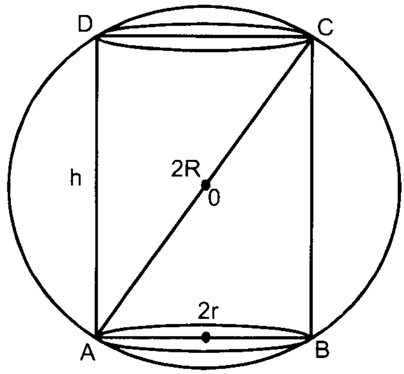
Then volume $V = \pi r^2 h$ ....(i)

We have $AC^2 = AB^2 + BC^2$

$$\Rightarrow \quad (2R)^2 = (2r)^2 + h^2$$

$$\Rightarrow \quad r^2 = \frac{1}{4}(4R^2 - h^2) \text{ ....(ii)}$$

By (i) and (ii) $V = \frac{\pi h}{4}(4R^2 - h^2)$



$$\therefore \quad \frac{dV}{dh} = \frac{\pi}{4}(4R^2 - h^2) + \frac{\pi h}{4}(-2h)$$

$$= \frac{\pi}{4}[4R^2 - 3h^2]$$

$$\therefore \quad \frac{d^2V}{dh^2} = \frac{\pi}{4}(-6h) = -\frac{3}{2}\pi h$$

For maxima or minima, $\frac{dV}{dh} = 0$

$$\text{i.e} \quad \frac{\pi}{4}(4R^2 - 3h^2) = 0$$

$$\Rightarrow \quad 4R^2 - 3h^2 = 0$$

$$\Rightarrow \quad 4R^2 = 3h^2 \Rightarrow h = \frac{2R}{\sqrt{3}}$$

When $h = \frac{2R}{\sqrt{3}}$, $\frac{d^2V}{dh^2} = -\frac{3}{2}\pi\frac{2R}{\sqrt{3}} = -\sqrt{3}\,\pi R < 0$

Hence volume of the cylinder is maximum when $h = \frac{2R}{\sqrt{3}}$

$$\text{Maximum volume} = \frac{\pi h}{4}(4R^2 - h^2)$$

$$= \frac{\pi}{4}\frac{2R}{\sqrt{3}}(4R^2 - \frac{4R^2}{3})$$

$$= \frac{4\pi R^2}{3\sqrt{3}}$$

Example 15. Show that of all rectangles inscribed in a given fixed circle, the square has the maximum area.

Solution: Let ABCD be a rectangle inscribed in a circle of radius r and centre at 0. Then

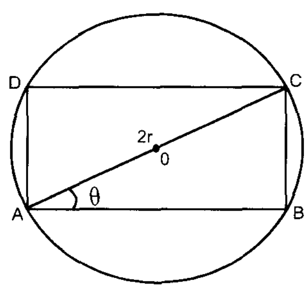
$AB = 2r\cos\theta$ and $BC = 2r\sin\theta$

Where $\theta = \angle BAC$

Let A be the area of the rectangle ABCD. Then

$A = AB \times BC = 4r^2\cos\theta\sin\theta = 2r^2\sin 2\theta$

$$\therefore \frac{dA}{d\theta} = 4r^2\cos 2\theta \text{ and } \frac{d^2A}{d\theta^2} = -8r^2\sin 2\theta$$



For maximum or minimum, $\frac{dA}{d\theta} = 0$

i.e $4r^2\cos 2\theta = 0$

$\Rightarrow \cos 2\theta = 0$

$\Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$

When $\theta = \frac{\pi}{4}$, $\frac{d^2A}{d\theta^2} = -8r^2\sin\frac{\pi}{2} = -8r^2 < 0$

Hence Area is maximum when $\theta = \frac{\pi}{4}$

Now when $\theta = \frac{\pi}{4}$ $AB = 2r\cos\frac{\pi}{4} = r\sqrt{2}$

$$BC = 2r \sin \frac{\pi}{4} = r\sqrt{2}$$

$\therefore AB = BC$

Therefore ABCD is a square

Exercises

1. Find for which values of x, the following functions are maximum and minimum.

   (i) $x^3 - 9x^2 + 15x - 3$ (ii) $4x^3 - 15x^2 + 12x - 2$

   (iii) $\dfrac{x^2 - 7x + 6}{x - 10}$ (iv) $\dfrac{x^2 + x + 1}{x^2 - x + 1}$

   (v) $x^4 - 8x^3 + 22x^2 - 24x + 5$

2. Show that the following function possess neither a maximum nor a minimum.

   (i) $x^3 - 3x^2 + 6x + 3$ (ii) $x^3 - 3x^2 + 9x - 1$

   (iii) $\dfrac{\sin(x + a)}{\sin(x + b)}$ (iv) $\dfrac{ax + b}{cx + d}$

3. (i) Show that the maximum value of $x + \dfrac{1}{x}$ is less than its minimum value

   (ii) Show that the minimum value of $\dfrac{(2x - 1)(x - 8)}{(x - 1)(x - 4)}$ is greater than its maximum value

4. Show that $x^3 - 6x^2 + 12x - 3$ is neither a maximum nor a minimum when $x=2$

5. Show that sinx (1+cox) is a maximum for $x = \pi/2$

6. Examine the maxima and minima of the following functions:

   (i) sinx (ii) x^6 (iii) x^5 (iv) cosx (v) e^xsinx

7. Show that $\sqrt{3}$ sinx + 3 cosx is a maximum for $x = \pi/6$

8. Show that $\sin^3x \cos x$ is a maximum when $x = \pi/3$

9. Show that

   (i) the minimum value of $\frac{x}{\log x}$ is e

   (ii) the minimum value of $4e^{2x} + e^{-3x}$ is 12

   (iii) $x^2 + x \sin x + 4\cos x$ is maximum for x=0 and minimum for $x = \pi/3$

   (iv) $4x - 8x\log_e^2$ is a mimimum when x=1

10. If f(x) = |x|, show that f(0) is a minimum although f′(0) does not exists

11. Find the point on the parabola $2y = x^2$ which is nearest to the point (0, 3)

12. If 40 square feet of a sheet of metal are to be used in the construction of an open tank with a square base, find the dimension so that the capacity is the greatest possible.

13. The sum of the surfaces of a cube and a sphere is given. Show that when the sum of their volumes is least, the diameter of the sphere is equal to the edge of the cube.

14. Show that the height of a closed cylinder of the given volume and least surface is equal to the diameter.

15. Show that the curved surface of a right circular cylinder of greatest curved surface which can be inscribed in a sphere is one half of that sphere.

16. Find the volume of the greatest cylinder that can be inscribed in a cone of height h and semi vertgical angle α.

17. Given the total surface of a right circular cone, show that when the volume of the cone is maximum, then the semi vertical angle will be $\sin^{-1}\left(\frac{1}{3}\right)$

18. Divide 80 into two parts such that the product of the cube of one and the 5th power of the other shall be as great as possible.

19. If the sum of the length of the hypotenuse and another side of a right-angle triangle is given, show that the area of the triangle is a maximum when the angle between them is $\pi/3$.

20. A gardener having 120 ft. of fencing wishes to enclose a rectangular plot of land and also to erect a fence across the land parallel to two sides. When is the maximum area he can enclose?

21. The intensity of light varies inversely as the square of the distance from the source. If two lights are 150ft apart and one light is 8 times as strong as the other, where should an object be placed between the lights to have the least illumination?

22. The force F exerted by a circular electric current of radius a on a magnet whose axis coicides with the axis of the coil is given by

$$F \alpha \; x\left(a^2 + x^2\right)^{-5/2}$$

wher x is the distance of the magnet from the centre of the circle. Show that F is greatest when $x = 9/2$

23. A window is in the form of a rectangle surmounted by a semi circle of the total perimeter be 25ft.; find the dimensions so that the greatest possible amount of light may be admitted.

24. A particle is moving in a straight line. Its distance x cm from a fixed point 0 at any time t seconds is given be the relation

$$x = t^4 - 10t^3 + 24t^2 + 36t + 12$$

When is it moving slowly?

10

Indeterminate Forms

Introduction

In general $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$

But if $\lim\limits_{x \to a} f(x) \to 0$ and $\lim\limits_{x \to a} g(x) \to 0$, then $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ reduces to the form $\dfrac{0}{0}$, which is meaningless and is called an indeterminate form. But is does not mean that its limit cannot be evaluated. In this chapter we shall discuss methods to evaluate such cases.

10.1 The form $\frac{0}{0}$ (L' Hospital's Theorem)

Let f(x) and g(x) be functions of x which can be expanded by Taylor's Theorem and also let f(a) = 0 = g(a). Then

$\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$ provided the latter limit exists.

Proof: By Taylor's Theorem we have

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f(a)+(x-a)f'(a)+\frac{(x-a)^2}{2!}f^2(a)+.......+R_1}{g(a)+(x-a)g'(a)+\frac{(x-a)^2}{2!}g^2(a)+.......+R_2}$$

Where $R_1 = \frac{(x-a)^n}{n!} f^n\{a+(x-a)\theta_1\}\ 0<\theta_1<1$

and $R_2 = \frac{(x-a)^n}{n!} f^n\{a+(x-a)\theta_2\}\ 0<\theta_2<1$

Since $f(a) = 0 = g(a)$

$$\therefore \quad \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{(x-a)f'(a)+\frac{(x-a)^2}{2!}f^2(a)+.......+R_1}{(x-a)g'(a)+\frac{(x-a)^2}{2!}g^2(a)+.......+R_2}$$

$$\Rightarrow \quad \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(a)+\frac{(x-a)}{2!}f^2(a)+.......}{g'(a)+\frac{(x-a)}{2!}g^2(a)+.......} = \frac{f'(a)}{g'(a)}$$

$$\therefore \quad \lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Remark: If $f'(a) = f^2(a) = f^3(a) = ...f^{n-1}(a) = 0$

and $g'(a) = g^2(a) = g^3(a) = ...g^{n-1}(a) = 0$

but $f^n(a)$ and $g^n(a)$ are not both zero. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f^n(x)}{g^n(x)}$$

The above proposition is true even when $n\to\infty$ instead of a

i.e $\lim_{x\to\infty} f(x) = 0 = \lim_{x\to\infty} g(x)$

$$\therefore \lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{t\to 0}\frac{f\left(\frac{1}{t}\right)}{g\left(\frac{1}{t}\right)} \quad \left[\text{putting } x = \frac{1}{t},\ t\to 0 \text{ as } x\to\infty\right]$$

$$= \lim_{t\to 0} \frac{f'\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)}{g'\left(\frac{1}{t}\right)\left(-\frac{1}{t^2}\right)}$$

$$= \lim_{t\to 0} \frac{f'\left(\frac{1}{t}\right)}{g'\left(\frac{1}{t}\right)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

10.2. The form $\frac{\infty}{\infty}$

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$. Then

$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, provided the limit exist

Proof: $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{1/g(x)}{1/f(x)} \to \frac{0}{0}$ form.

$$= \lim_{x\to a} \frac{-g'(x)/[g(x)]^2}{-f'(x)/[f(x)]^2} \text{ by L Hospital Rule}$$

$$\Rightarrow \quad \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \left[\frac{g'(x)}{f'(x)} \cdot \left\{\frac{f(x)}{g(x)}\right\}^2\right]$$

$$= \lim_{x\to a} \frac{g'(x)}{f'(x)} \cdot \lim_{x\to a} \left\{\frac{f(x)}{g(x)}\right\}^2 \quad \text{..........(A)}$$

Let $\lambda = \lim_{x\to a} \frac{f(x)}{g(x)}$ .....(i)

Then 3 cases arise

Case 1. If $\lambda \neq 0$

In the case dividing both sides of (A) by λ^2 we get

$$\frac{1}{\lambda} = \lim_{x \to a} \frac{g'(x)}{f'(x)}$$

$$\Rightarrow \quad \lambda = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Hence $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ by (i)

Case 2. If $\lambda=0$

In this case adding 1 to both sides of (1) we get

$$\lambda + 1 = \lim_{x \to a} \frac{f(x)}{g(x)} + 1 = \lim_{x \to a} \frac{f(x)+g(x)}{g(x)}$$

$$\Rightarrow \quad \lambda + 1 = \lim_{x \to a} \frac{f'(x)+g'(x)}{g'(x)} \text{ by case 1}$$

i.e $\quad \lambda + 1 = \frac{f'(x)}{g'(x)} + 1$

$$\Rightarrow \quad \lambda = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

$$\Rightarrow \quad \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \text{ by (i)}$$

Case 3. If $\lambda = \infty$

Then $\lim_{x \to a} \frac{1}{f(x)/g(x)} = \lim_{x \to a} \frac{g(x)}{f(x)} = \lim_{x \to a} \frac{g'(x)}{f'(x)}$ by case 2

$$\Rightarrow \quad \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Hence in every case, If $\lim_{x \to a} f(x) = \infty = \lim_{x \to a} g(x)$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

10.3 The form $0\times\infty$

Thus form can be changed either to $\frac{0}{0}$ form or $\frac{\infty}{\infty}$ form and then we can proceed as above

Let $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = \infty$. Then

$$\lim_{x\to a} f(x).g(x) = \lim_{x\to a} \frac{f(x)}{1/g(x)} \quad [\tfrac{0}{0} \text{ form}]$$

or $$\lim_{x\to a} f(x).g(x) = \lim_{x\to a} \frac{g(x)}{1/f(x)} \quad [\tfrac{\infty}{\infty} \text{ form}]$$

10.4 The form $\infty-\infty$

Let $\lim\limits_{x\to a} f(x) = \infty = \lim\limits_{x\to a} g(x)$, then

$$\lim_{x\to a}[f(x) - g(x)] = \lim_{x\to a}\left[\frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x)g(x)}}\right]$$

Which is of the form $\frac{0}{0}$ and can be evaluated as in 10.1

10.5 The forms 0^0, ∞^0, 1^∞

A function of the form u^v when u and v are functions of x, can be written as

$$u^v = e^{v\log u}$$

Hence the function u^v becomes indeterminate when the exponent of e i.e vlogu takes the form ∞ and this will be the case when

(i) $v = 0$, $u = 0$ for $\log 0 = -\infty$

(ii) $v = 0$, $u = \infty$ for $\log\infty = \infty$

(iii) $v = \infty$, $u = 1$ for $\log_e^1 = 0$

Now

(i) When $v = 0$, $u = 0$, the function assumes the form 0^0

(ii) When $v = 0$, $u = \infty$, the function assumes the form ∞^0

(iii) When $v = \infty$, $u = 1$, the function assumes the form 1^∞

Thus we see that the function which takes the forms 0^0, ∞^0 and 1^∞ are indeterminate.

10.6 Working Rules

(A) If $\frac{f(x)}{g(x)}$ assumes the form $\frac{0}{0}$, when x=a, find the limiting value as 'x' approaches 'a' indefinitely take the derivative of the numerator divide it by the derivative of the denominator and put x=a in the result.

If $\frac{f'(a)}{g'(a)}$ assummes the form $\frac{0}{0}$ again, then the same rule is to be applied to the function, so that its limit, as $x \to a$ is $\frac{f^2(a)}{g^2(a)}$ or $\frac{f''(a)}{g''(a)}$ and so for similar cases. Several repetitions of the process are sometimes necessary before the value of the function can be ascertained.

(B) For finding the limit of a fraction assuming the form $\frac{\infty}{\infty}$ for x=a we arrive at the same rule as for fractions that assume the form $\frac{0}{0}$ for x=a.

The process may be repeated several times in succession if each new fraction we get goes on assuming the form $\frac{\infty}{\infty}$ or $\frac{0}{0}$.

(C) In evaluating the limits of the functions which take the forms 0^0, ∞^0, 1^∞ we denote the function by y; take logarithms of both sides to the base 'e' and find the limiting value of right side which takes the form 0, ∞ and find logy. Now equate y to 'e' raised to this limiting value as power.

Illustrative Examples

1. Evaluate $\lim_{x \to 0} \frac{\sin x}{x}$

Solution: $\lim_{x \to 0} \frac{\sin x}{x} \to \quad [\frac{0}{0} \text{ form}]$

$$= \lim_{x \to 0} \frac{\cos x}{1} \text{ by L'Hospital's Rule}$$

$$= \frac{1}{1} = 1$$

2. Evaluate $\lim_{x \to 0} \left(\frac{\tan x}{x}\right)$

Solution: $\lim_{x \to 0} \frac{\tan x}{x} \rightarrow [\frac{0}{0} \text{ form}]$

$$= \lim_{x \to 0} \frac{\sec^2 x}{1} \text{ by L'Hospital's Rule}$$

$$= \frac{1}{1} = 1$$

3. Evaluate $\lim_{x \to 1} \frac{\log x}{x - 1}$

Solution: $\lim_{x \to 1} \frac{\log x}{x - 1} \rightarrow [\frac{0}{0} \text{ form}]$

$$= \lim_{x \to 1} \frac{1/x}{1} \text{ by L'Hospital's Rule}$$

$$= \frac{1}{1} = 1$$

4. Evaluate $\lim_{x \to 0} \frac{\log(1 - x^2)}{\log \cos x}$

Solution: $\lim_{x \to 0} \frac{\log(1 - x^2)}{\log \cos x} \rightarrow [\frac{0}{0} \text{ form}]$

$$= \lim_{x \to 0} \frac{\frac{-2x}{1 - x^2}}{\frac{-\sin x}{\cos x}}$$

$$= \lim_{x \to 0} \frac{2x}{(1-x^2)\tan x} \rightarrow \quad [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \frac{2}{(1-x^2)\sec^2 x - 2x\tan x} = \frac{2}{1} = 2$$

5. Evaluate $\lim\limits_{x \to 0} \dfrac{(1+x)^n - 1}{x}$

Solution: $\lim\limits_{x \to 0} \dfrac{(1+x)^n - 1}{x} \rightarrow \quad [\dfrac{0}{0} \text{ form}]$

$$= \lim_{x \to 0} \frac{n(1+x)^{n-1}}{1} = n$$

6. Evaluate $\lim\limits_{x \to 0} \dfrac{1-\cos x}{x^2}$

Solution: $\lim\limits_{x \to 0} \dfrac{1-\cos x}{x^2} \rightarrow \quad [\dfrac{0}{0} \text{ form}]$

$$= \lim_{x \to 0} \frac{\sin x}{2x} \rightarrow \quad [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \frac{\cos x}{2} \text{ by L'Hospital's Rule}$$

$$= \frac{1}{2}$$

7. Evaluate $\lim\limits_{x \to 0} \dfrac{x - \tan x}{x^3}$

Solution: $\lim\limits_{x \to 0} \dfrac{x - \tan x}{x^3} \rightarrow \quad [\dfrac{0}{0} \text{ form}]$

$$= \lim_{x \to 0} \frac{1 - \sec^2 x}{3x^2} \rightarrow \quad [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \frac{-2\sec^2 x \tan x}{6x} \rightarrow \quad [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \frac{-2(\sec^4 x + 2\sec^2 x \tan^2 x)}{6}$$

$$= \frac{-2}{6} = -\frac{1}{3}$$

8. Evaluate $\lim_{x \to 0} \frac{xe^x - \log(1+x)}{x^2}$

Solution: $\lim_{x \to 0} \frac{xe^x - \log(1+x)}{x^2} \to [\frac{0}{0}$ form]

$$= \lim_{x \to 0} \frac{xe^x + e^x - \frac{1}{1+x}}{2x} \to [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \frac{xe^x + e^x + e^x + (1+x)^{-2}}{2}$$

$$= \frac{0+1+1+1}{2} = \frac{3}{2}$$

9. Evaluate $\lim_{x \to 0} \frac{e^x + e^{-x} - 2}{x^2}$

Solution: $\lim_{x \to 0} \frac{e^x + e^{-x} - 2}{x^2} \to [\frac{0}{0}$ form]

$$= \lim_{x \to 0} \frac{e^x - e^{-x}}{2x} \to [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \frac{e^x + e^{-x}}{2}$$

$$= \frac{1+1}{2} = \frac{2}{2} = 1$$

10. Evaluate $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$

Soluion: $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x} \to [\frac{0}{0}$ form]

$$= \lim_{x\to 0} \frac{e^x - \cos x\, e^{\sin x}}{1-\cos x} \rightarrow [\tfrac{0}{0} \text{ form}]$$

$$= \lim_{x\to 0} \frac{e^x - \cos^2 e^{\sin x} + \sin x\, e^{\sin x}}{\sin x} \rightarrow [\tfrac{0}{0} \text{ form}]$$

$$= \lim_{x\to 0} \frac{e^x - \cos^3 x\, e^{\sin x} + 3\sin x \cos x\, e^{\sin x} + \cos x e^{\sin x} + \cos x \sin x\, e^{\sin x}}{\cos x}$$

$$= \frac{1-1+0+1+0}{1} = 1$$

11. Evaluate $\lim_{x\to 1}\left[\frac{1}{\log x} - \frac{x}{\log x}\right]$

Solution: $\lim_{x\to 1}\left[\frac{1}{\log x} - \frac{x}{\log x}\right]$

$$= \lim_{x\to 1} \frac{1-x}{\log x} \rightarrow [\tfrac{0}{0} \text{ form}]$$

$$= \lim_{x\to 1} \frac{-1}{1/x} = -1$$

12. Evaluate $\lim_{x\to 0} \frac{x\cos x - \log x(1+x)}{x^2}$

Solution: $\lim_{x\to 0} \frac{x\cos x - \log x(1+x)}{x^2} \rightarrow [\tfrac{0}{0} \text{ form}]$

$$= \lim_{x\to 0} \frac{\cos x - x\sin x - \frac{1}{1+x}}{2x} \rightarrow [\tfrac{0}{0} \text{ form}]$$

$$= \lim_{x\to 0} \frac{-\sin x - \sin x - x\cos x + (1+x)^{-2}}{2} = \frac{1}{2}$$

13. Evaluate $\lim_{x\to 0} \frac{a^x - 1 - x\log a}{x^2}$

Solution: $\lim_{x\to 0} \frac{a^x - 1 - x\log a}{x^2} \rightarrow [\tfrac{0}{0} \text{ form}]$

$$= \lim_{x \to 0} \frac{a^x \log a - \log a}{2x} \quad \to \quad [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \frac{a^x (\log a)^2}{2} = \frac{(\log a)^2}{2}$$

14. Evaluate $\lim_{x \to 0} \frac{\log \sin 2x}{\log \sin x}$

Solution: $\lim_{x \to 0} \frac{\log \sin 2x}{\log \sin x} \quad \to \quad [\frac{\infty}{\infty} \text{ form}]$

$$= \lim_{x \to 0} \frac{\frac{1}{\sin 2x} 2\cos 2x}{\frac{1}{\sin x} \cos x}$$

$$= \lim_{x \to 0} \frac{2\cot 2x}{\cot x} \quad \to \quad [\frac{\infty}{\infty} \text{ form}]$$

$$= \lim_{x \to 0} \frac{2\tan x}{\tan 2x} \quad \to \quad [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \frac{2\sec x}{2\sin^2 2x} \text{ by L'Hospital's Rule}$$

$$= 1$$

15. Evaluate $\lim_{x \to \pi/2} \frac{\tan x}{\tan 3x}$

Solution: $\lim_{x \to \pi/2} \frac{\tan x}{\tan 3x} \quad \to \quad [\frac{\infty}{\infty} \text{ form}]$

$$= \lim_{x \to \pi/2} \frac{\sec^2 x}{3\sec^2 3x} \quad \to \quad [\frac{\infty}{\infty} \text{ form}]$$

$$= \lim_{x \to \pi/2} \frac{\cos^2 3x}{3\cos^2 x} \quad \to \quad [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to \pi/2} \frac{-6\cos 3x \sin 3x}{-6\cos x \sin x}$$

$$= \lim_{x \to \pi/2} \frac{\sin 6x}{\sin 2x} \to [\frac{0}{0} \text{ form}] \; [\because \; \sin 2x = 2\sin x \cos x]$$

$$= \lim_{x \to \pi/2} \frac{6\cos 6x}{2\cos 2x} = 3$$

16. Evaluate $\lim_{x \to \infty} \dfrac{\log(1+x)}{x}$

Solution: $\lim_{x \to \infty} \dfrac{\log(1+x)}{x} \to [\frac{\infty}{\infty} \text{ form}]$

$$= \lim_{x \to \infty} \frac{\frac{1}{x}}{1}$$

$$= \lim_{x \to \infty} \frac{1}{1+x} = 0$$

17. Evaluate $\lim_{x \to 1} \dfrac{\log\cos\left(\frac{\pi x}{2}\right)}{\log(1-x)}$

Solution: $\lim_{x \to 1} \dfrac{\log\cos\left(\frac{\pi x}{2}\right)}{\log(1-x)} \to [\frac{\infty}{\infty} \text{ form}]$

$$= \lim_{x \to 1} \frac{\frac{1}{\cos\frac{\pi x}{2}}\left(-\frac{\pi}{2}\sin\frac{\pi x}{2}\right)}{-\frac{1}{1-x}}$$

$$= \lim_{x \to 1} \frac{\tan\frac{\pi x}{2}.\frac{\pi}{2}}{\frac{1}{1-x}} \to [\frac{\infty}{\infty} \text{ form}]$$

$$= \lim_{x \to 1} \frac{\frac{\pi}{2}(1-x)}{\cot\frac{\pi x}{2}} \to [\frac{0}{0} \text{ form}]$$

$$= \lim_{x\to 1} \frac{\frac{\pi}{2}(-1)}{-\frac{\pi}{2}\operatorname{cosec}^2\frac{\pi x}{2}} = 1$$

18. Evaluate $\lim_{x\to \pi/2} \frac{\log\left(x-\pi/2\right)}{\tan x}$

Solution: $\lim_{x\to \pi/2} \frac{\log\left(x-\pi/2\right)}{\tan x} \to [\frac{\infty}{\infty}$ form]

$$= \lim_{x\to \pi/2} \frac{\frac{1}{x-\pi/2}}{\sec^2 x} \to [\frac{\infty}{\infty} \text{ form}]$$

$$= \lim_{x\to \pi/2} \frac{\cos^2 x}{x-\pi/2} \to [\frac{0}{0} \text{ form}]$$

$$= \lim_{x\to \pi/2} \frac{-2\cos x \sin x}{1} = 0$$

19. Evaluate $\lim_{x\to 0} \log_{\tan x} \tan 2x$

Solution: $\lim_{x\to 0} \log_{\tan x} \tan 2x$

$$= \lim_{x\to 0} \frac{\log_e \tan 2x}{\log_e \tan x} \to [\frac{\infty}{\infty} \text{ form}] \quad \left[\because \log_b^a = \frac{\log_e^a}{\log_e^b}\right]$$

$$= \lim_{x\to 0} \frac{\frac{1}{\tan 2x} 2\sec^2 2x}{\frac{1}{\tan x}\sec^2 x} \to [\frac{\infty}{\infty} \text{ form}]$$

$$= \lim_{x\to 0} \frac{2\sec^2 2x \tan x}{\tan 2x \sec^2 x}$$

$$= \lim_{x\to 0} \frac{2\sec^2 2x}{\sec^2 x} \times \lim_{x\to 0} \frac{\tan x}{\tan 2x} \to \text{(i)}$$

$$= 2\times\lim_{x\to 0}\frac{\tan x}{\tan 2x} \rightarrow [\frac{0}{0} \text{ form}]$$

$$= 2\times\lim_{x\to 0}\frac{\sec^2 x}{2\sec^2 2x}$$

$$= 2\times\frac{1}{2} = 1$$

20. Evaluate $\lim_{x\to 0} \log_{\sin x}(\sin 2x)$

Solution: Same as Ex. 19

21. Evaluate $\lim_{x\to\infty} x^n e^{-x}$ where $n\in IN$

Solution: $\lim_{x\to\infty} x^n e^{-x} = \lim_{x\to\infty}\frac{x^n}{e^n} \rightarrow [\frac{\infty}{\infty} \text{ form}]$

$$= \lim_{x\to\infty}\frac{n x^{n-1}}{e^x} \rightarrow [\frac{\infty}{\infty} \text{ form}]$$

$$= \lim_{x\to\infty}\frac{n(n-1)x^{n-2}}{e^x} \rightarrow [\frac{\infty}{\infty} \text{ form}]$$

$$= \lim_{x\to\infty}\frac{n!}{e^x} = 0$$

22. Evaluate $\lim_{x\to 0}\sin x \log x$

Solution: $\lim_{x\to 0}\sin x \log x \rightarrow (0\times\infty \text{ form})$

$$= \lim_{x\to 0}\frac{\log x}{1/\sin x} \rightarrow [\frac{\infty}{\infty} \text{ form}]$$

$$= \lim_{x\to 0}\frac{\frac{1}{x}}{-\operatorname{cosec} x \cot x} \qquad \left[\because \frac{1}{\sin x}\ \operatorname{cosec} x\right]$$

$$= \lim_{x\to 0} -\frac{\sin x \tan x}{x} \rightarrow [\frac{0}{0} \text{ form}]$$

$$= \lim_{x\to 0}\frac{-\sin x \sec^2 x + \tan x \cos x}{1} = 0$$

23. Evaluate $\lim_{x \to \pi/2} (1-\sin x)\tan x$

Solution: $\lim_{x \to \pi/2} (1-\sin x)\tan x \to (0\times\infty \text{ form})$

$$= \lim_{x \to \pi/2} \frac{1-\sin x}{\cot x} \to [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to \pi/2} \frac{-\cos x}{-\operatorname{cosec}^2 x} = \lim_{x \to \pi/2} \cos x \sin^2 x = 0$$

24. Evaluate $\lim_{x \to 1} \sec\frac{\pi}{2x} \log x$

Solution: $\lim_{x \to 1} \sec\frac{\pi}{2x} \log x \to (0\times\infty \text{ form})$

$$= \lim_{x \to 1} \frac{\log x}{\cos\frac{\pi}{2x}} \to [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 1} \frac{1/x}{\frac{\pi}{2}\sin\frac{\pi}{2x}} = \frac{2}{\pi}$$

25. Evaluate $\lim_{x \to \infty} 2^x \sin\left(\frac{a}{2^x}\right)$

Solution: $\lim_{x \to \infty} 2^x \sin\left(\frac{a}{2^x}\right) \to (0\times\infty \text{ form})$

$$= \lim_{x \to \infty} \frac{\sin\left(a/2^x\right)}{1/2^x} \to [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to \infty} a \frac{\sin\left(a/2^x\right)}{\left(a/2^x\right)}$$

$$= \lim_{y \to 0} a \frac{\sin y}{y}$$ [putting $y = \frac{a}{2^x}$ when $x \to \infty$, $y \to \infty$]

$$= a\times 1 = a$$ $\left[\because \lim_{y \to 0} \frac{\sin y}{y} = 1\right]$

26. Evaluate $\lim\limits_{x \to \infty} \left(a^{1/x} - 1\right)x$

Solution: $\lim\limits_{x \to \infty} \left(a^{1/x} - 1\right)x \rightarrow (0 \times \infty \text{ form})$

$$= \lim_{x \to \infty} \frac{a^{1/x} - 1}{1/x} \rightarrow [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to \infty} \frac{a^{1/x} \log a \times \left(-\frac{1}{x^2}\right)}{\left(-\frac{1}{x^2}\right)}$$

$$= \lim_{x \to \infty} a^{1/x} \log a = a^0 \log a = \log a$$

27. Evaluate $\lim\limits_{x \to \pi/2} (\sec x - \tan x)$

Solution: $\lim\limits_{x \to \pi/2} (\sec x - \tan x) \rightarrow (0 \times \infty \text{ form})$

$$= \lim_{x \to \pi/2} \left[\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right]$$

$$= \lim_{x \to \pi/2} \frac{1 - \sin x}{\cos x} \rightarrow [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to \pi/2} \frac{-\cos x}{-\sin x}$$

$$= \lim_{x \to \pi/2} \cot x = 0$$

28. Evaluate: $\lim\limits_{x \to 0} \frac{1}{x} - \cot x$

Solution: $\lim\limits_{x \to 0} \frac{1}{x} - \cot x \rightarrow (\infty - \infty)$

$$= \lim_{x \to 0} \left[\frac{1}{x} - \frac{\cos x}{\sin x}\right]$$

$$= \lim_{x \to 0} \left[\frac{\sin x - x\cos x}{x\sin x}\right] \to [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \frac{\cos x - \cos x + x\sin x}{\sin x + x\cos x}$$

$$= \lim_{x \to 0} \frac{x\sin x}{\sin x + x\cos x} \to [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \frac{\sin x + x\cos x}{2\cos x - x\sin x} = \frac{0}{2} = 0$$

29. Evaluate $\lim_{x \to \pi/2} \left(\sec x - \frac{1}{1 - \sin x}\right)$

Solution: $\lim_{x \to \pi/2} \left(\sec x - \frac{1}{1 - \sin x}\right) \to (\infty - \infty \text{ form})$

$$= \lim_{x \to \pi/2} \left(\frac{1}{\cos x} - \frac{1}{1 - \sin x}\right)$$

$$= \lim_{x \to \pi/2} \left[\frac{1}{\cos x} - \frac{1}{1 - \sin x}\right]$$

$$= \lim_{x \to \pi/2} \frac{1 - \sin x - \cos x}{\cos x(1 - \sin x)} \to [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to \pi/2} \frac{\sin x - \cos x}{-\sin x(1 - \sin x) + \cos x(1 - \cos x)}$$

$$= \lim_{x \to \pi/2} \frac{\sin x - \cos x}{-\sin x + \sin^2 x + \cos x - \cos^2 x}$$

$$= \frac{1}{-1 + 1} = \infty$$

30. Evaluate $\lim_{x \to 1} \left[\frac{2}{x^2 - 1} - \frac{1}{x - 1}\right]$

Solution: $\lim_{x \to 1} \left[\frac{2}{x^2 - 1} - \frac{1}{x - 1}\right] \to (\infty - \infty \text{ form})$

$$= \lim_{x \to 1} \left[\frac{2-(x+1)}{x^2-1}\right]$$

$$= \lim_{x \to 1} \frac{1-x}{x^2-1} = -\frac{1}{2}$$

31. Evaluate $\lim_{x \to 0} \left(\frac{1}{x^2} - \cot^2 x\right)$

Solution: $\lim_{x \to 0} \left(\frac{1}{x^2} - \cot^2 x\right) \to (\infty - \infty \text{ form})$

$$= \lim_{x \to 0} \left[\frac{1}{x^2} - \frac{\cos^2 x}{\sin^2 x}\right]$$

$$= \lim_{x \to 0} \left[\frac{\sin^2 x - x^2 \cos^2 x}{x^2 \sin^2 x}\right] \to [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \left[\frac{2\sin x \cos x - 2x\cos^2 x - 2x^2 \cos x \sin x}{2x\sin^2 x + x^2 2\sin x \cos x}\right] \to [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \frac{2\cos 2x - 2\cos^2 x - 2x.2\cos x \sin x - 4x\cos x \sin x - x^2 2\cos 2x}{2x\sin^2 x + 2x.2\sin x \cos x + 2x\sin 2x + x^2 2\cos 2x} \to [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \frac{-4\sin 2x + 4\cos x \sin x - 2\sin 2x - 2x2\cos 2x - 2\sin x - 2x2\cos 2x - 4x\cos 2x + 2x^2 2\sin x}{4\sin x \cos x + 2\sin 2x + 2x2\cos 2x + 2\sin 2x + 2x2\cos 2x + 4x\cos 2x - 4x^2 \sin 2x}$$

$$= \lim_{x \to 0} \frac{-8\sin 2x - 12x\cos 2x + 2\sin 2x + 4x^2 \sin x}{6\sin 2x + 12x\cos 2x - x2\cos 2x + 4x^2 \sin 2x} \to [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \frac{-16\cos 2x - 12\cos 2x + 24x\sin 2x + 4\cos 2x + 8x\sin 2x + 8x^2 \cos 2x}{12\cos 2x + 12\cos 2x - 24x\sin 2x - 8x\sin 2x - 8x^2 \cos 2x}$$

$$= \frac{-16-12+4}{12+12} = \frac{-24}{24} = -1$$

32. Evaluate $\lim_{x \to 0} \left\{\frac{1}{x} - \frac{1}{x^2}\log(1+x)\right\}$

Solution: $\lim_{x \to 0} \left\{\frac{1}{x} - \frac{1}{x^2}\log(1+x)\right\} \to (\infty - \infty \text{ form})$

$$= \lim_{x \to 0} \left\{ \frac{x - \log(1+x)}{x^2} \right\} \to [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \left\{ \frac{1 - \frac{1}{1+x}}{2x} \right\} \to [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \frac{(1+x)^{-2}}{2} = \frac{1}{2}$$

33. Evaluate $\lim_{x \to 0} \left[\frac{1}{e^x - 1} - \frac{1}{x} \right]$

Solution: $\lim_{x \to 0} \left[\frac{1}{e^x - 1} - \frac{1}{x} \right] \to (\infty - \infty \text{ form})$

$$= \lim_{x \to 0} \frac{x - e^x + 1}{x(e^x - 1)} \to [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \frac{1 - e^x}{(e^x - 1) + x.e^x} \to [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \frac{-e^x}{e^x + e^x + x e^x}$$

$$= -\frac{1}{1+1} = -\frac{1}{2}$$

34. Evaluate $\lim_{x \to 0} x^x$

Solution: $\lim_{x \to 0} x^x \to [0^0 \text{ form}]$

Let $y = \lim_{x \to 0} x^x$ Then

$\log y = \lim_{x \to 0} x \log x \to [0 \times \infty \text{ form}]$

$\Rightarrow \log y = \lim_{x \to 0} \frac{\log x}{1/x} \to [\frac{\infty}{\infty} \text{ form}]$

$$\Rightarrow \quad \log y = \lim_{x\to 0} \frac{1/x}{\left(-1/x^2\right)} = \lim_{x\to 0} (-x)$$

$$\Rightarrow \quad \log y = 0$$

$$\Rightarrow \quad y = e^0 = 1$$

$$\Rightarrow \quad \lim_{x\to 0} x^x = 1$$

35. Evaluate the following limit using L'Hospital's rule

(i) $\lim_{x\to \pi/2} \left(x - \frac{\pi}{2}\right) \sec x$ (ii) $\lim_{x\to 0} \frac{\log \tan 7x}{\log \tan 2x}$ (NEHU 2009)

Solution: $\lim_{x\to \pi/2} \left(x - \frac{\pi}{2}\right) \sec x \rightarrow [0\times\infty \text{ form}]$

$$= \lim_{x\to \pi/2} \frac{x - \pi/2}{\cos x} \rightarrow [\tfrac{0}{0} \text{ form}]$$

$$= \lim_{x\to \pi/2} \frac{1}{-\sin x} = -1$$

(ii) $\lim_{x\to 0} \frac{\log \tan 7x}{\log \tan 2x} \rightarrow [\frac{\infty}{\infty} \text{ form}]$

$$= \lim_{x\to 0} \frac{\frac{1}{\tan 7x} 7\sec^2 7x}{\frac{1}{\tan 2x} 2\sec^2 2x}$$

$$= \frac{7}{2} \lim_{x\to 0} \frac{\tan 2x}{\tan 7x} \frac{\sec^2 7x}{\sec^2 2x}$$

$$= \frac{7}{2} \lim_{x\to 0} \frac{\tan 2x}{\tan 7x} \times \lim_{x\to 0} \frac{\sec^2 7x}{\sec^2 2x}$$

$$= \frac{7}{2} \lim_{x\to 0} \frac{\tan 2x}{\tan 7x} \times 1$$

$$= \frac{7}{2} \times \lim_{x\to 0} \frac{\tan 2x}{\tan 7x} \rightarrow [\tfrac{0}{0} \text{ form}]$$

$$= \frac{7}{2} \times \lim_{x \to 0} \frac{2\sec^2 2x}{7\sec^2 7x}$$

$$= \frac{7}{2} \times \frac{2}{7} \lim_{x \to 0} \frac{\sec^2 2x}{\sec^2 7x}$$

$$= 1$$

36. Evaluate $\lim_{x \to 0} \left[\frac{1}{x^2} - \frac{1}{\sin^2 x}\right]$ [NEHU 2013]

Solution: $\lim_{x \to 0} \left[\frac{1}{x^2} - \frac{1}{\sin^2 x}\right] \to (\infty - \infty \text{ form})$

$$= \lim_{x \to 0} \left[\frac{\sin^2 x - x^2}{x^2 \sin^2 x}\right] \to [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \left[\frac{2\sin x \cos x - 2x}{2x^2 \sin x \cos x + 2x \sin^2 x}\right] \to [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \left[\frac{2\cos 2x - 2}{2x^2 \cos 2x + 2x \sin 2x + 2\sin^2 x + 2x\, 2\sin x \cos x}\right]$$

$$= \lim_{x \to 0} \left[\frac{2\cos 2x - 2}{2x^2 \cos 2x + 4x \sin 2x + 2\sin^2 x}\right] \to [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \left[\frac{-4\sin 2x}{-4x^2 \sin 2x + 4x \cos 2x + 4\sin 2x + 8x \cos 2x + 4\sin x \cos x}\right]$$

$$= \lim_{x \to 0} \left[\frac{-4\sin 2x}{-4x^2 \sin 2x + 4x \cos 2x + 4\sin 2x + 8x \cos 2x + 2\sin 2x}\right] \to [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \frac{-8\cos 2x}{-8x \sin 2x - 8x^2 \cos 2x + 4\cos 2x - 8x \sin 2x + 8\cos 2x + 8\cos 2x - 16x \sin 2x + 4\cos 2x}$$

$$= \frac{-8}{4 + 8 + 8 + 4}$$

$$= \frac{-8}{24} = -\frac{1}{3}$$

37. Use L'Hospital's rule to evaluate

$$\lim_{x\to 0}\frac{\tan nx - n\tan x}{n\sin x - \sin nx}$$ (NEHU 2008, 2005)

Solution: $\lim_{x\to 0}\frac{\tan nx - n\tan x}{n\sin x - \sin nx} \to [\frac{0}{0}\text{ form}]$

$$= \lim_{x\to 0}\frac{n\sec^2 nx - n\sec^2 x}{n\cos x - n\cos nx} \to [\frac{0}{0}\text{ form}]$$

$$= \lim_{x\to 0}\frac{n\,2\sec nx\sec nx\tan x.n - n\,2\sec x\sec x\tan x}{-n\sin x + n^2\sin nx}$$

$$= \lim_{x\to 0}\frac{2n^2\sec^2 nx\tan nx - 2n\sec^2 x\tan x}{n^2\sin nx - n\sin x}$$

$$= \lim_{x\to 0}\frac{2n^2\left(2n\sec^2 nx\tan^2 nx + n\sec^4 nx\right) - 2n\left(2\sec^2 x\tan^2 x + \sec^4 x\right)}{n^3\cos nx - n\cos x}$$

$$= \frac{2n^2(n) - 2n(1)}{n^3 - n}$$

$$= \frac{2n(n^2-1)}{n(n^2-1)} = 2$$

38. Evaluate $\lim_{x\to 1} x^{1/x-1}$ (NEHU 2014)

Solution: $\lim_{x\to 1} x^{1/x-1} \to [1^\infty\text{ form}]$

Let $y = \lim_{x\to 1} x^{1/x-1}$

$$\therefore \log y = \lim_{x\to 1}\frac{1}{x-1}\log x \to [\frac{0}{0}\text{ form}]$$

$$= \lim_{x\to 1}\frac{1/x}{1}$$

$$= \lim_{x\to 1}\frac{1}{x} = 1$$

$\therefore\ y = e^1 = e$

i.e $\lim_{x \to 1} x^{1/(x-1)} = e$

39. Evaluate $\lim_{x \to 0} \dfrac{e^x + \sin x - 1}{\log(1+x)}$ (NEHU 2007)

Solution: $\lim_{x \to 0} \dfrac{e^x + \sin x - 1}{\log(1+x)} \to [\frac{0}{0} \text{ form}]$

$$= \lim_{x \to 0} \frac{e^x + \cos x}{\frac{1}{1+x}} = \frac{2}{\frac{1}{1}} = 2$$

40. Use L'Hospital's rule to evaluate $\lim_{x \to 0} \dfrac{\log(x^2)}{\log(\cot^2 x)}$ (NEHU 2006)

Solution: $\lim_{x \to 0} \dfrac{\log(x^2)}{\log(\cot^2 x)} \to [\frac{\infty}{\infty} \text{ form}]$

$$= \lim_{x \to 0} \frac{\frac{2x}{x^2}}{\frac{1}{\cot^2 x} 2\cot x\left(-\operatorname{cosec}^2 x\right)}$$

$$= \lim_{x \to 0} \frac{2\cot x}{-x \operatorname{cosec}^2 x}$$

$$= \lim_{x \to 0} - \frac{2\cos x / \sin x}{x \, 1/\sin^2 x}$$

$$= \lim_{x \to 0} - \frac{2\sin x \cos x}{x}$$

$$= \lim_{x \to 0} - \frac{\sin 2x}{x} \to [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} - \frac{2\cos 2x}{1} = -2$$

41. Using L'Hospital's rule evaluate

(i) $\lim_{x\to e}(\log x)^{\frac{1}{1-\log x}}$ (ii) $\lim_{x\to 0}\sin x.\log x^2$ (NEHU 2010)

Solution: (i) $\lim_{x\to e}(\log x)^{\frac{1}{1-\log x}} \to [1^{\infty}$ form]

Let $y = \lim_{x\to e}(\log x)^{\frac{1}{1-\log x}}$

$\therefore \log y = \lim_{x\to e}\frac{1}{1-\log x}\log(\log x) \to [\frac{0}{0}$ form]

$$= \lim_{x\to e}\frac{\frac{1}{\log x}.\frac{1}{x}}{-\frac{1}{x}}$$

$$= \lim_{x\to e} -\frac{1}{\log x} = -1$$

$\therefore y = e^{-1} = \frac{1}{e}$

i.e $\lim_{x\to e}(\log x)^{\frac{1}{1-\log x}} = \frac{1}{e}$

(ii) $\lim_{x\to 0}\sin x.\log x^2 \to [0\times\infty$ form]

$= \lim_{x\to 0}\frac{\log x^2}{\operatorname{cosec} x} \to [\frac{\infty}{\infty}$ form]

$$= \lim_{x\to 0}\frac{\frac{1}{x^2}2x}{-\operatorname{cosec} x\cot x}$$

$$= 2\lim_{x\to 0} -\frac{1}{x^2\operatorname{cosec} x\cot x}$$

$= 2\lim_{x\to 0} -\frac{\sin^2 x}{x^2\cos x} \to [\frac{0}{0}$ form]

$$= -2 \lim_{x \to 0} \frac{2 \sin x \cos x}{2x \cos x - x^2 \sin x} \to [\frac{0}{0} \text{ form}]$$

$$= -2 \lim_{x \to 0} \frac{\sin 2x}{2x \cos x - x^2 \sin x}$$

$$= -2 \lim_{x \to 0} \frac{2 \cos 2x}{2 \cos x - 2x \sin x - 2x \sin x - x^2 \cos x}$$

$$= -2 \times \frac{2}{2-0-0-0} = -2$$

42. Prove that $\lim_{x \to 0} \frac{\log(1+x)}{x} = 1$. Use this result to show that

$$\lim_{x \to 0} \frac{\log(1+8x)}{\log(1+7x)} = \frac{8}{7}$$ (NEHU 2000)

Solution: $\lim_{x \to 0} \frac{\log(1+x)}{x} \to [\frac{0}{0} \text{ form}]$

$$= \lim_{x \to 0} \frac{\frac{1}{1+x}}{1} = 1$$

Now $\lim_{x \to 0} \frac{\log(1+8x)}{\log(1+7x)} = \lim_{x \to 0} \frac{\frac{\log(1+8x)}{8x}.8}{\frac{\log(1+7x)}{7x}.7}$

$$= \frac{8}{7} \lim_{x \to 0} \frac{\frac{\log(1+8x)}{8x}}{\frac{\log(1+7x)}{7x}}$$

$$= \frac{8}{7} \times \frac{1}{1} \text{ (from above)}$$

$$= \frac{8}{7}$$

43. Evaluate $\lim\limits_{x \to 0} (\cos x)^{1/x^2}$

Solution: $\lim\limits_{x \to 0} (\cos x)^{1/x^2} \rightarrow [1^{\infty} \text{ form}]$

Let $y = \lim\limits_{x \to 0} (\cos x)^{1/x^2}$

$$\Rightarrow \log y = \lim_{x \to 0} \frac{\log \cos x}{x^2} \rightarrow [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \frac{\frac{1}{\cos x}(-\sin x)}{2x}$$

$$= \lim_{x \to 0} \frac{-\tan x}{2x} \rightarrow [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \frac{-\sec^2 x}{2} = -\frac{1}{2}$$

$\therefore y = e^{1/2}$

$$\Rightarrow \lim_{x \to 0} (\cos x)^{1/x^2} = e^{-1/2}$$

44. Evaluate $\lim\limits_{x \to \pi/2} (\cos x)^{\cos x}$

Solution: $\lim\limits_{x \to \pi/2} (\cos x)^{\cos x} \rightarrow [0^0 \text{ form}]$

Let $y = \lim\limits_{x \to \pi/2} (\cos x)^{\cos x}$

$$\Rightarrow \log y = \lim_{x \to \pi/2} \cos x \log (\cos x) \rightarrow [0 \times \infty \text{ form}]$$

$$\Rightarrow \log y = \lim_{x \to \pi/2} \frac{\log (\cos x)}{\sec x} \rightarrow [\frac{\infty}{\infty} \text{ form}]$$

$$= \lim_{x \to \pi/2} \frac{\frac{1}{\cos x}(-\sin x)}{\sec x \tan x}$$

$$= \lim_{x \to \pi/2} -\frac{\sin x}{\tan x}$$

$$= \lim_{x \to \pi/2} -\cos x = 0$$

$\therefore y = e^0 = 1$

$\Rightarrow \lim_{x \to \pi/2} (\cos x)^{\cos x} = 1$

45. Evaluate $\lim_{x \to 0} (\sin x)^{\tan x}$

Solution: $\lim_{x \to 0} (\sin x)^{\tan x} \to [0^0 \text{ form}]$

Let $y = \lim_{x \to 0} (\sin x)^{\tan x}$

$\therefore \log y = \lim_{x \to 0} \tan x \log (\sin x) \to [0 \times \infty \text{ form}]$

$$\Rightarrow \log y = \lim_{x \to 0} \frac{\log(\sin x)}{\cot x} \to [\frac{\infty}{\infty} \text{ form}]$$

$$= \lim_{x \to 0} \frac{\frac{1}{\sin x} \cos x}{-\operatorname{cosec}^2 x}$$

$$= \lim_{x \to 0} \frac{\frac{\cos x}{\sin x}}{-\frac{1}{\sin x}}$$

$$= \lim_{x \to 0} (-\sin x \cos x) = 0$$

$\Rightarrow y = e^0 = 1$

$\therefore \lim_{x \to 0} (\sin x)^{\tan x} = 1$

46. Evaluate $\lim_{x \to \pi/2} (\tan x)^{\cos x}$

Solution: $\lim_{x \to \pi/2} (\tan x)^{\cos x} \to [\infty^0 \text{ form}]$

Let $y = \lim_{x \to \pi/2} (\tan x)^{\cos x}$

$\Rightarrow \log y = \lim_{x \to \pi/2} \cos x \log (\tan x) \to [0 \times \infty \text{ form}]$

$$= \lim_{x \to \pi/2} \frac{\log \tan x}{\sec x} \to [\frac{\infty}{\infty} \text{ form}]$$

$$\Rightarrow \log y = \lim_{x \to \pi/2} \frac{\frac{1}{\tan x} \sec^2 x}{\sec x \tan x}$$

$$= \lim_{x \to \pi/2} \frac{\sec x}{\tan^2 x} \rightarrow [\frac{\infty}{\infty} \text{ form}]$$

$$= \lim_{x \to \pi/2} \frac{1/\cos x}{\sin^2 x / \cos^2 x}$$

$$= \lim_{x \to \pi/2} \frac{\cos x}{\sin^2 x} = \frac{0}{1} = 0$$

$\therefore$ $y = e^0 = 1$

$$\Rightarrow \lim_{x \to 0} (\tan x)^{\cos x} = 1$$

47. Evaluate $\lim_{x \to 0} (\cos x)^{\cot^2 x}$

Solution: $\lim_{x \to 0} (\cos x)^{\cot^2 x} \rightarrow [1^\infty \text{ form}]$

Let $y = \lim_{x \to 0} (\cos x)^{\cot^2 x}$

$$\Rightarrow \log y = \lim_{x \to 0} \cot^2 x \log(\cos x) \rightarrow [0 \times \infty \text{ form}]$$

$$= \lim_{x \to 0} \frac{\log(\cos x)}{\tan^2 x} \rightarrow [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to 0} \frac{\frac{1}{\cos x}(-\sin x)}{2 \tan x \sec^2 x}$$

$$= \lim_{x \to 0} -\frac{1}{2\sec^2 x} = -\frac{1}{2}$$

$\therefore$ $y = e^{1/2}$

$$\Rightarrow \lim_{x \to 0} (\cos x)^{\cot^2 x} = e^{1/2}$$

48. Evaluate $\lim_{x \to \pi/2} (\sin x)^{\tan x}$

Solution: $\lim_{x \to \pi/2} (\sin x)^{\tan x} \rightarrow [1^\infty \text{ form}]$

Let $y = \lim_{x \to \pi/2} (\sin x)^{\tan x}$

$\Rightarrow \log y = \lim_{x \to \pi/2} \tan \log(\sin x) \rightarrow$ [$0 \times \infty$ form]

$$= \lim_{x \to \pi/2} \frac{\log \sin x}{\cot x} \rightarrow [\frac{0}{0} \text{ form}]$$

$$= \lim_{x \to \pi/2} \frac{\frac{1}{\sin x} \cos x}{-\operatorname{cosec}^2 x}$$

$$= \lim_{x \to \pi/2} \frac{\cos x / \sin x}{-1/\sin^2 x}$$

$$= \lim_{x \to \pi/2} (-\sin x \cos x) = 0$$

$\therefore y = e^0 = 1$

$\Rightarrow \lim_{x \to \pi/2} (\sin x)^{\tan x} = 1$

Exercises

1. Evaluate thefollowing limits (Ex. 1 to 3):

(i) $\lim_{x \to a} \frac{x^n - a^n}{x - a}$ (ii) $\lim_{x \to 0} \frac{e^{2x} - 1}{\log(1 + x)}$

(iii) $\lim_{x \to 0} \frac{x - \sin^{-1} x}{\sin^3 x}$ (iv) $\lim_{x \to 2} \frac{x^3 - 2x^2 + 2x - 4}{x^2 - 5x + 6}$

(v) $\lim_{x \to 0} \frac{e^x + \sin x - 1}{\log(1 + x)}$ (vi) $\lim_{x \to 0} \frac{\sin \log(1 + x)}{\log(1 + \sin x)}$

(vii) $\lim_{x \to 0} \frac{\log x^2}{\log(\cot^2 x)}$ (viii) $\lim_{x \to 0} \frac{e^x - e^{\sin x}}{x - \sin x}$

2. (i) $\lim_{x \to \pi/2} \frac{\tan 5x}{\tan x}$ (ii) $\lim_{x \to 0} x \log \sin^2 x$

(iii) $\lim_{x \to \pi/2} \sec 5x \cos 7x$ (iv) $\lim_{x \to \infty} \frac{x^2 + 3x}{1 - 5x^2}$

(v) $\lim_{x \to 0} \sin x \log x^2$ (vi) $\lim_{x \to 0} \log_{\tan^2 x} (\tan^2 2x)$

(vii) $\lim_{x \to \pi/2} \sec x \left(x \sin x - \frac{\pi}{2}\right)$ (viii) $\lim_{x \to 0} \left(\frac{1}{x} - \cot x\right)$

3. (i) $\lim_{x \to e} (\log x)^{\frac{1}{1-\log x}}$ (ii) $\lim_{x \to 0} x^{2x}$

(iii) $\lim_{x \to 0} x^{2\sin x}$ (iv) $\lim_{x \to 1} x^{\frac{1}{1-x}}$

4. Evaluate the following limits.

(i) $\lim_{x \to 0} \frac{x \cos x - \log(1+x)}{x^2}$ (ii) $\lim_{x \to \pi/2} \left(x - \frac{\pi}{2}\right) \sec x$

(iii) $\lim_{x \to 0} \left[\frac{1}{x} - \frac{1}{x^2}\log(1+x)\right]$ (iv) $\lim_{x \to 0} \frac{\tan x \tan^{-1} x - x^2}{x^6}$

(v) $\lim_{x \to 0} \frac{\log \tan 7x}{\log \tan 2x}$ (vi) $\lim_{x \to 0} \frac{x - \sin x \cos x}{x^3}$

(vii) $\lim_{x \to \pi/2} \sin x^{\tan x}$

5. Evaluate $\lim_{x \to 2} \left[\frac{1}{\log(x-1)} - \frac{1}{x-2}\right]$

6. Evaluate $\lim_{x \to 1} \left[\frac{x}{x-1} - \frac{1}{\log x}\right]$

7. Evaluate $\lim_{x \to 0} (\cot^2 x)^{\sin x}$

8. Evaluate $\lim_{x \to 0} \left[\frac{1}{x^2} - \frac{1}{\sin^2 x}\right]$

9. Show that $\lim_{x \to 0} \frac{e^x - 1}{x^4 \sin x} \left[\frac{3\sin x - \sin 3x}{\cos x - \cos 3x}\right]^4$

10. Show that $\lim_{x \to 0} \frac{\log_{\sin^2 x}(\cos x)}{\log_{\sin^2 x}(\cos \frac{1}{2} x)} = 4$

11. Evaluate $\lim_{x \to 0} \frac{\tan nx - n \tan x}{n \sin x - \sin nx}$

11

Tangents and Normals

Introduction

In the previous chapter we have explained that $\frac{dy}{dx}$ at the point P of the curve $y = f(x)$ represents geometrically the gradient at the point P i.e. $\frac{dy}{dx}$ represents the trigonometrical tangent of the angle of inclination which the tangent at P makes with the positive direction of x-axis.

In otherwords the derivative at any point on a curve is the slope of the tangent at the point (x, y) on the curve.

i.e $\frac{dy}{dx} = \tan \psi$ = slope of the tangent

The following should be carefully noted:

In a curve $y = f(x)$, if at a particular point (x, y)

(i) $\frac{dy}{dx} = 1$, then the slope of the tangent at that point is 1, i.e. the tangent at that point of the curve makes an angle of 45^0 with the positive direction of x-axis.

(ii) $\frac{dy}{dx} = 0$, then the tangent to the curve at that point makes an angle of 0^0 with the positive direction of x axis i.e. the tangent to the curve at that point is parallel to x-axis.

(iii) $\frac{dy}{dx}$ = negative, then the tangent to the curve at that point makes an obtuse angle with the positive direction of x-axis.

(iv) $\frac{dy}{dx} = \infty$, then the tangent to the curve at that point makes an angle of 90^0 with the positive direction of x-axis i.e. the tangent to the curve at that point is perpendicular to x-axis

In Coordinate Geometry, the coordinates of the fixed point is generally represented by x and y. Thus the equation of the straight line passing through the point (x_1, y_1) and making an angle ψ with the positive direction of x-axis is given by

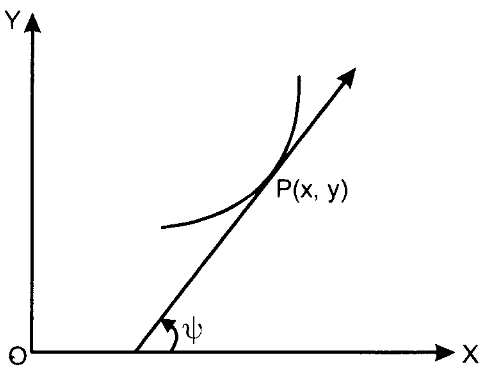
$y - y_1 = m\ (x - x_1)$ where $m = \tan\psi$

In the application of Differential Calculus to the theory of plane curves, the symbols X and Y are used to denote the current coordinates in the equations of the tangents and normal to a curve. The symbols x and y are used as usual for the current coordinates in the equation of the curve.

11.1 Equation of Tangent at any point (x, y) of the curve y = f(x)

Let P (x, y) be any point on the curve y = f(x)

The equation of any straight line passing throught the point P(x, y) is $Y - y = m\ (X - x)$



Where $m = \tan\psi$, ψ being the angle of inclination of this line with the positive direction of x-axis.

If this line is to be a tangent to the curve y = f(x) at the point P(x,y) then

$\tan\psi = \frac{dy}{dx}$.

Hence the equation of the tangent at the point P(x, y) is

$$Y - y = \frac{dy}{dx}(X-x)$$

11.2 Equation of the tangent at any point on the curve f(x, y) = 0

If the equation of the curve is of the form f(x, y) = 0, then

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y} \qquad \because f(x, y) = 0$$

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial x}dy = 0$$

Equation of the tangent at the point (x, y) is

$$Y-y = -\frac{\partial f/\partial x}{\partial f/\partial y}(X - x)$$

$$\Rightarrow \quad (X-x)\frac{\partial f}{\partial x} + (Y - y)\frac{\partial f}{\partial y} = 0 \qquad \frac{\partial f}{\partial y} = f_x$$

$$\text{or} \quad (X - x) f_x + (Y - y) f_y = 0 \qquad \frac{\partial f}{\partial x} = f_x$$

11.3 Parametric Form

If the equation of the curve is of the form

x = f(t) y = ϕ(t), then

$$\frac{dx}{dt} = f'(t) \quad \frac{dy}{dt} = \phi'(t)$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\phi'(t)}{f'(t)}$$

Hence the equation of the tangent is

$$Y - \phi(t) = \frac{\phi'(t)}{f'(t)}(X - f(t))$$

11.4 Definition

A normal to the curve is a straight line perpendicular to the tengent at the point of contact.

11.5 Equation of the Normal at any point (x, y) of the curve y = f(x)

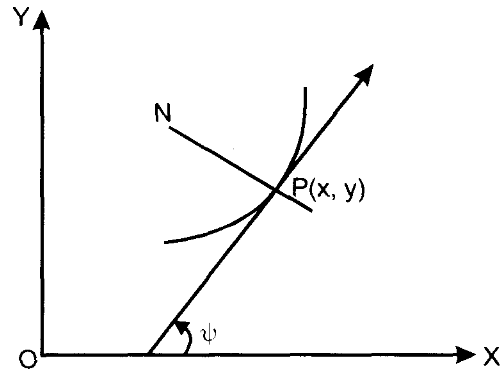
The equation of the tangent to the curve y = f(x) at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x) \quad ....(i)$$

Its gradient $= \frac{dy}{dx}$

Any line throught (x, y) is given by

$$Y - y = m\,(X - x) \quad ....(ii)$$



If the line (2) is normal to the curve, then it is perpendicular to the tangent (i)

i.e $\quad m.\frac{dy}{dx} = -1$

$\Rightarrow \quad m = -\frac{dx}{dy}$

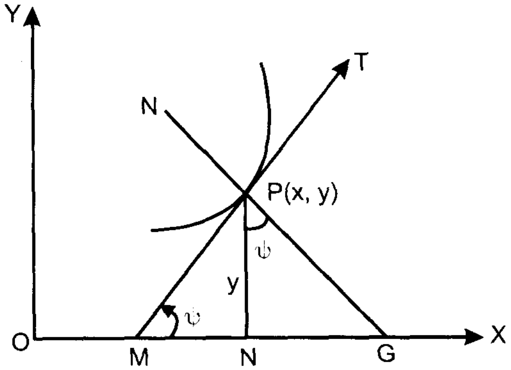
∴ From (2), the equation of the normal to the curve y = f(x) at the point (x, y) is

$$Y - y = -\frac{dx}{dy}(X - x)$$

or $\quad Y - y = -\frac{1}{dy/dx}(X - x)$

11.6 Cartesian Sub-tangent, Sub-normal, Lengths of Tangents and Normals

Let P(x, y) be any point on the curve y = f(x).



Let the tangent PT and the normal PN to the curve at P meet X axis at M and G respectively. Then MN is called the subtangent, NG is called the sub-normal and the lengths of tangent intercepted between the point of contact and the x-axis i.e PM is called the length of the tangent. Similarly the length of the normal intercepted between the point of contact and the x-axis i.e PG is called the length of the normal. i.e

(i) Subtangent = MN = $y \cot\psi = \dfrac{y}{\tan\psi} = \dfrac{y}{dx/dy} = y\dfrac{dx}{dy}$

(ii) Sub-normal = NG = $y \tan\psi = y.\dfrac{dy}{dx}$

(iii) Length of tangent = PM = $y \operatorname{cosec}\psi = y\sqrt{1+\cot^2\psi}$

$$= y\frac{\sqrt{1+\tan^2\psi}}{\tan\psi}$$

$$= y\frac{\sqrt{1+\left(dy/dx\right)^2}}{dy/dx}$$

(iv) Length of Normal = PG = $y \sec\psi = y\sqrt{1+\tan^2\psi}$

$$= y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Illustrative Examples

Example 1. Find the equation of the tangent at any point of the parabola y2 = 4ax

Solution: Given curve is $y^2 = 4ax$

differentiating both sides w.r.t x

$$2y \frac{dy}{dx} = 4a$$

$$\text{i.e } \frac{dy}{dx} = \frac{2a}{y}$$

Equation of the tangent is

$$Y - y = \frac{dy}{dx} (X - x)$$

$$\text{i.e} \quad Y - y = \frac{2a}{y} (X - x)$$

$$\Rightarrow \quad Yy - y^2 = 2aX - 2ax$$

$$\Rightarrow \quad Yy - 4ax = 2aX - 2ax$$

$$\Rightarrow \quad Yy = 2aX + 2ax$$

i.e $Yy = 2a (X + x)$

Example 2. Find the equation of the tangent at any point (x, y) of the ellipe

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$$

Solution: In this case $f(x, y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$

$$\therefore \frac{\partial f}{\partial x} = \frac{2x}{a^2}, \frac{\partial f}{\partial y} = \frac{2y}{b^2}$$

$\therefore$ Equation of the tangent at any point (x, y) is

$$(X - x) \frac{\partial f}{\partial x} + (Y - y) \frac{\partial f}{\partial y} = 0$$

$$\text{i.e} \quad (X - x) \frac{2x}{a^2} + (Y - y) \frac{2y}{b^2} = 0$$

$$\Rightarrow \quad \frac{Xx}{a^2} - \frac{x^2}{a^2} + \frac{Yy}{b^2} - \frac{y^2}{b^2} = 0$$

$$\Rightarrow \quad \frac{Xx}{a^2} + \frac{Yy}{b^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$\therefore \quad \frac{Xx}{a^2} + \frac{Yy}{b^2} = 1 \qquad\qquad \because \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Example 3. Find the equation of the tangent at the point 't' on the curve given by $x = a\cos^3 t$, $y = a\sin^3 t$

Solution: Here $x = a\cos^3 t$, $y = a\sin^3 t$

$$\therefore \quad \frac{dx}{dt} = -3a\cos^2 t \sin t$$

$$\frac{dy}{dt} = 3a\sin^2 t \cos t$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3a\sin^2 t\cos t}{-3a\cos^2 t\sin t} = -\frac{\sin t}{\cos t}$$

$\therefore$ Equation of the tangent is $Y - y = \frac{dy}{dx}(X - x)$

$$\Rightarrow \quad Y - a\sin^3 t = - \; -\frac{\sin t}{\cos t}(X - a\cos^3 t)$$

$$\Rightarrow \quad Y\cos t - a\sin^3 t\cos t = -X\sin t + a\cos^3 t\sin t$$

$$\Rightarrow \quad X\sin t + Y\cos t = a\sin t\cos^3 t + a\sin^3 t + \cos t$$

$$= a\sin t\cos t(\cos^2 t + \sin^2 t)$$

i.e $\quad X\sin t + Y\cos t = a\sin t\cos t$

or $\quad X\sec t + Y\operatorname{cosec} t = a$

Example 4. Find the equation of the tangent of (x, y) to the curve $\frac{x^m}{a^m} + \frac{y^m}{b^m} - 1$

Solution: Here $f(x, y) = \frac{x^m}{a^m} + \frac{y^m}{b^m} - 1 = 0$

$$\therefore \frac{\partial f}{\partial x} = \frac{m x^{m-1}}{a^m}, \frac{\partial f}{\partial y} = \frac{m y^{m-1}}{b^m}$$

Equation of the tangent at (x, y) is

$$(Y - y) \frac{\partial f}{\partial y} + (X - x) \frac{\partial f}{\partial x} = 0$$

i.e $$(Y - y) \frac{m y^{m-1}}{b^m} + (X - x) \frac{m x^{m-1}}{a^m} = 0$$

$$\Rightarrow \quad mX \frac{x^{m-1}}{a^m} + mY \frac{y^{m-1}}{b^m} = m \frac{x^m}{a^m} + m \frac{y^m}{b^m}$$

i.e $$\frac{X x^{m-1}}{a^m} + \frac{Y y^{m-1}}{b^m} = 1$$

Example 5. If p = x cosα + y sinα touches the curve $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$, then prove that $p^{m/(m-n)} = (a\cos\alpha)^{m/(m-n)} + (b\sin\alpha)^{m/(m-n)}$

Solution: Here Given curve $\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1$

$$\therefore \frac{\partial f}{\partial x} = \frac{m x^{m-1}}{a^m}, \frac{\partial f}{\partial y} = \frac{m y^{m-1}}{b^m}$$

Equation of the tangent at (x, y) is

$$(X - x) \frac{\partial f}{\partial x} + (Y - y) \frac{\partial f}{\partial y} = 0$$

i.e $$(X - x) \frac{m x^{m-1}}{a^m} + (Y - y) \frac{m y^{m-1}}{b^m} = 0$$

$$\Rightarrow \quad \frac{X x^{m-1}}{a^m} + \frac{Y y^{m-1}}{b^m} = 1 \quad .....(1)$$ [as in previous example]

If p = x cosα + y sinα touches the given curve

then these two equations are identical

$$\therefore \quad \frac{x^{m-1}/a^m}{\cos\alpha} = \frac{y^{m-1}/b^m}{\sin\alpha} = \frac{1}{p}$$

$$\therefore \quad \frac{x^{m-1}}{a^m} = \frac{\cos\alpha}{p} \text{ and } \frac{y^{m-1}}{b^m} = \frac{\sin\alpha}{p}$$

$$\therefore \quad \left(\frac{x}{a}\right)^{m-1} = \frac{a\cos\alpha}{p} \text{ and } \left(\frac{y}{b}\right)^{m-1} = \frac{a\sin\alpha}{p}$$

$$\Rightarrow \quad \left\{\left(\frac{x}{a}\right)^{m-1}\right\}^{\frac{m}{m-1}} = \left(\frac{a\cos\alpha}{p}\right)^{\frac{m}{m-1}} \text{ and } \left\{\left(\frac{y}{b}\right)^{m-1}\right\}^{\frac{m}{m-1}} = \left(\frac{b\sin\alpha}{p}\right)^{\frac{m}{m-1}}$$

$$\therefore \quad \left(\frac{x}{a}\right)^{m} = \left(\frac{a\cos\alpha}{p}\right)^{\frac{m}{(m-1)}}, \quad \left(\frac{y}{b}\right)^{m} = \left(\frac{b\sin\alpha}{p}\right)^{\frac{m}{(m-1)}}$$

$$\therefore \quad \left(\frac{a\cos\alpha}{p}\right)^{\frac{m}{(m-1)}} + \left(\frac{b\sin\alpha}{p}\right)^{\frac{m}{(m-1)}} = \left(\frac{x}{a}\right)^{m} + \left(\frac{y}{b}\right)^{m}$$

$$\Rightarrow \quad \left(\frac{a\cos\alpha}{p}\right)^{\frac{m}{(m-1)}} + \left(\frac{b\sin\alpha}{p}\right)^{\frac{m}{(m-1)}} = 1$$

$$\Rightarrow \quad p^{m/(m-1)} = (a\cos\alpha)^{m/(m-1)} + (b\sin\alpha)^{m/(m-1)}$$

Example 6. If $p = x\cos\alpha + y\sin\alpha$ touch the curve

$\left(\frac{x}{a}\right)^{n-1} + \left(\frac{y}{b}\right)^{\frac{n}{n-1}} = 1$, then prove that $p^n = (a\cos\alpha)^n + (b\sin\alpha)^n$

Solution: Same as above

Example 7. Prove that all points on the curve $y^2 = 4a\left\{x + a\sin\frac{x}{a}\right\}$ at which the tangent is parallel to the x-axis be on a parabola.

Solution: Given curve is $y^2 = 4a\left\{x + a\sin\frac{x}{a}\right\}$ ....(i)

$$\therefore \quad 2y\frac{dy}{dx} = 4a\{1 + a.\frac{1}{a}\cos\frac{x}{a}\}$$

$$\Rightarrow \quad y\frac{dy}{dx} = 2a\{1 + \cos\frac{x}{a}\}$$

Since the tangent is parallel to the x-axis

$$\therefore \quad \frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2a\{1 + \cos\frac{x}{a}\} = 0$$

$$\Rightarrow \quad 1 + \cos\frac{x}{a} = 0$$

$$\Rightarrow \quad \cos\frac{x}{a} = -1$$

$$\Rightarrow \quad \cos^2\frac{x}{a} = 1$$

$$\Rightarrow \quad \sin^2\frac{x}{a} = 1 - \cos^2\frac{x}{a}$$

$$\Rightarrow \quad \sin^2\frac{x}{a} = 1 - 1 = 0$$

$$\Rightarrow \quad \sin\frac{x}{a} = 0$$

Hence from (1) we get $y^2 = 4ax$ which is a qarabola. Hence the proof.

Example 8. Prove that $\frac{x}{a}+\frac{y}{b}=1$ touches the curve $y = be^{-x/a}$ at the point where the curve crosses the axis of y.

Solution: Given curve is $y = be^{-x/a}$ .... (1)

$$\therefore \quad \frac{dy}{dx} = -\frac{b}{a}e^{-x/a}$$

At the point where the curve crosses the y-axis, x=0

$$\therefore \frac{dy}{dx} = -\frac{b}{a}$$

$\therefore$ Equation of the tangent at this point $(0, y_1)$ is

$$y - y_1 = -\frac{b}{a}(x - 0)$$

$$\Rightarrow \quad ay - ay_1 = -bx$$

$$\Rightarrow \quad \frac{y}{b} - \frac{y_1}{b} = -\frac{x}{a} \text{ (dividing by ab)}$$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} = \frac{y_1}{b} \quad \text{......... (i)}$$

Siince this point $(0, y_1)$ lies on the curve (i)

$$\therefore y_1 = be^0 = b \Rightarrow \frac{y_1}{b} = 1$$

$\therefore$ from (i) $\frac{x}{a} + \frac{y}{b} = 1$ is the required tangent to the given curve (i)

Hence $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point where it croses y-axis

Example 9. If the line lx + my = 1 touches the curve $(ax)^n + (by)^n = 1$ show that

$$\left(\frac{1}{m}\right)^{n/n-1} + \left(\frac{m}{b}\right)^{n/n-1} = 1$$

Solution: Let the line lx + my = 1 touches the curve $(ax)^n + (by)^n = 1$ at (x_1, y_1). Then

$$lx_1 + my_1 = 1 \quad \text{.........(i)}$$

and $(ax_1)^n + (by_1)^n = 1$ ....(ii)

Now given curve is $(ax)^n + (by)^n = 1$

$$\therefore \quad a^n.n\, x^{n-1} \frac{dy}{dx} = -n\, a^n\, x^{n-1}$$

$$\Rightarrow \quad b^n ny^{n-1} \frac{dy}{dx} = -n a^n x^{n-1}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{na^n x^{n-1}}{nb^n y^{n-1}} = -\frac{a^n x^{n-1}}{b^n y^{n-1}}$$

at (x_1, y_1), $\frac{dy}{dx} = -\frac{a^n x_1^{n-1}}{b^n y_1^{n-1}}$

$\therefore$ Equation of the tangent at (x_1, y_1) is

$$y - y_1 = --\frac{a^n x_1^{n-1}}{b^n y_1^{n-1}} (x - x_1)$$

$$\Rightarrow \quad b^n y_1^{n-1} y - b^n y_1^n = -a^n x_1^{n-1} x - a^n x_1^n$$

$$\Rightarrow \quad a^n x_1^{n-1} x + b^n y_1^n y = a^n x_1^{n-1} + b^n y_1^n$$

$$\Rightarrow \quad a^n x_1^{n-1} x + b^n y_1^{n-1} y = 1 \quad \text{..........(iii) by (ii)}$$

Since this line is a tangent to the given curve $(ax)^n + (by)^n = 1$, it must be identical with the given line $lx + my = 1$

$$\therefore \quad \frac{a^n x_1^{n-1}}{l} = \frac{b^n y_1^{n-1}}{m} = 1$$

$$\Rightarrow \quad \frac{l}{a} = a^{n-1} x^{n-1} \text{ and } \frac{m}{b} = b^{n-1} y^{n-1}$$

$$\Rightarrow \quad \left(\frac{l}{a}\right) = (ax)^{n-1} \text{ and } \left(\frac{m}{b}\right) = (by)^{n-1}$$

$$\Rightarrow \quad \left(\frac{l}{a}\right)^{1/n-1} = ax \text{ and } \left(\frac{m}{b}\right)^{1/n-1} = by$$

$$\Rightarrow \quad \left(\frac{l}{a}\right)^{1/n-1} = (ax)^n \text{ and } \left(\frac{m}{b}\right)^{1/n-1} = (by)^n$$

$$\therefore \quad \left(\frac{l}{a}\right)^{1/n-1} + \left(\frac{m}{b}\right)^{1/n-1} = (ax)^n + (by)^n = 1$$

Example 10. If the line $x\cos\alpha + y\sin\alpha = p$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, show that $(a\cos\alpha)^2 + (b\sin\alpha)^2 = p^2$

Solution: Suppose the line $x\cos\alpha + y\sin\alpha = p$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point $(x_1, y_1,)$

Then $x_1\cos\alpha + y_1\sin\alpha = p$ ....(i)

and $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$ ....(ii)

Now Given curve is $f(x, y) = \frac{x^2}{a^2}+\frac{y^2}{b^2} - 1 = 0$

$$\therefore \frac{\partial f}{\partial x} = \frac{2x}{a^2} \text{ and } \frac{\partial f}{\partial y} = \frac{2y}{b^2}$$

$\therefore$ Equation of the tangent at (x_1, y_1) is

$$(x - x_1)\frac{\partial f}{\partial x} + (y - y_1)\frac{\partial f}{\partial y} = 0$$

$$\Rightarrow (x - x_1)\frac{2x}{a^2} + (y - y_1)\frac{2y}{b^2} = 0$$

$$\Rightarrow \frac{x^2}{a^2} - \frac{xx_1}{a^2} + \frac{y^2}{b^2} - \frac{yy_1}{b^2} = 0$$

$$\Rightarrow \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$\Rightarrow \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \text{................(iii)}$$

Since this line (iii) is a tangent to the given ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, it must be indentical with the line $x\cos\alpha + y\sin\alpha = p$

$$\therefore \frac{x_1/a^2}{\cos\alpha} = \frac{y_1/b^2}{\sin\alpha} = \frac{1}{p}$$

$$\Rightarrow \quad \frac{\cos\alpha}{p} = \frac{x_1}{a^2} \text{ and } \frac{\sin\alpha}{p} = \frac{y_1}{b^2}$$

$$\Rightarrow \quad \frac{a\cos\alpha}{p} = \frac{x_1}{a} \text{ and } \frac{b\sin\alpha}{p} = \frac{y_1}{b}$$

$$\Rightarrow \quad \left(\frac{a\cos\alpha}{p}\right)^2 = \frac{x_1^{\ 2}}{a^2} \text{ and } \left(\frac{b\sin\alpha}{p}\right)^2 = \frac{y_1^{\ 2}}{b^2}$$

$$\therefore \quad \left(\frac{b\cos\alpha}{p}\right)^2 + \left(\frac{b\sin\alpha}{p}\right)^2 = \frac{x_1^{\ 2}}{a^2} + \frac{y_1^{\ 2}}{b^2}$$

$$\Rightarrow \quad \frac{(a\cos\alpha)^2}{p^2} + \frac{(b\sin\alpha)^2}{p^2} = 1 \text{ by (ii)}$$

$$\Rightarrow \quad (a\cos\alpha)^2 + (b\sin\alpha)^2 = p^2$$

Example 11. If the line $x\cos\alpha + y\sin\alpha = p$ touches the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, prove that $16\cos^2\alpha + 9\sin^2\alpha = p^2$

Solution: Same as above

Example 12. Prove that in the curve $y = b\,e^{x/a}$, then sub-tangent is constant and sub normal is y^2/a

Solution: The given curve is $y = b\,e^{x/a}$\

$$\therefore \quad \frac{dy}{dx} = \frac{b}{a}\,e^{x/a}$$

$$\therefore \quad \frac{dx}{dy} = \frac{a}{b}\,e^{-x/a}$$

$$\text{Sub-tangent} = y\,\frac{dx}{dy} = b\,e^{x/a}.\frac{a}{b}\,e^{-x/a}$$

$$= a = \text{constant}$$

$$\text{Sub-normal} = y\,\frac{dy}{dx} = b\,e^{x/a}.\frac{b}{a}\,e^{x/a}$$

$$= \frac{b^2 e^{2x/a}}{a} = \frac{\left(b\,e^{x/a}\right)^2}{a} = \frac{y^2}{a}$$

Example 13. Show that in any curve

$$\frac{\text{sub} - \text{normal}}{\text{sub tangent}} = \left[\frac{\text{length of normal}}{\text{length of tangent}}\right]^2$$

Solution: $$\frac{\text{Length of normal}}{\text{Length of tangent}} = \frac{y\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{y\frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{\frac{dy}{dx}}} = \frac{dy}{dx}$$

$$\therefore \left[\frac{\text{length of normal}}{\text{length of tangent}}\right]^2 = \left(\frac{dy}{dx}\right)^2 \quad \text{........(A)}$$

Again $$\frac{\text{Sub normal}}{\text{Sub tangent}} = \frac{y\frac{dy}{dx}}{y\frac{dx}{dy}} = \left(\frac{dy}{dx}\right)^2 \quad \text{.........(B)}$$

By (A) and (B) we get

$$\frac{\text{Sub normal}}{\text{Sub tangent}} = \left[\frac{\text{length of normal}}{\text{length of tangent}}\right]^2$$

Example 14. Find the lengths of the subtangent, subnormal, tangent and normal to the curve $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, at the point θ

Solution: Given curve is $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$

$$\therefore \quad \frac{dx}{d\theta} = a\ (1 + \cos\theta)\ ; \quad \frac{dy}{d\theta} = a(\sin\theta)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}$$

$$\therefore \quad \frac{dy}{dx} = \tan\ \theta/2$$

$$\therefore \quad \text{Subtangent} = y\frac{dy}{dx} = a(1 - \cos\theta)\ \cot\theta/2$$

$$= a\ 2\sin^2 \theta/2 \ \frac{\cos \theta/2}{\sin \theta/2}$$

$$= a\ 2\sin \theta/2 \cos \theta/2$$

$$= a \sin \theta$$

$$\text{Subnormal} = y\frac{dy}{dx} = a(1 - \cos\theta)\tan \theta/2$$

$$= a\ 2\sin^2 \theta/2 \tan \theta/2$$

$$= 2a\sin^2 \theta/2 \tan \theta/2$$

$$\text{Length of tangent} = \frac{y\sqrt{1+\left(dy/dx\right)^2}}{dy/dx}$$

$$= \frac{a(1-\cos\theta)\sqrt{1+\tan^2 \theta/2}}{\tan \theta/2}$$

$$= \frac{a\,2\sin^2 \theta/2 \sec \theta/2}{\tan \theta/2}$$

$$= \frac{2a\sin^2 \theta/2}{\cos \theta/2} \times \frac{\cos \theta/2}{\sin \theta/2}$$

$$= 2a\sin \theta/2$$

$$\text{Length of normal} = y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$= a\ (1-\cos\theta)\ \sqrt{1+\tan^2 \theta/2}$$

$$= a.2\sin^2 \theta/2 \sec \theta/2 = 2a\sin \theta/2 \tan \theta/2$$

Example 15. Find the lengths of the sub tangent, sub normal, tangent and normal to the curve

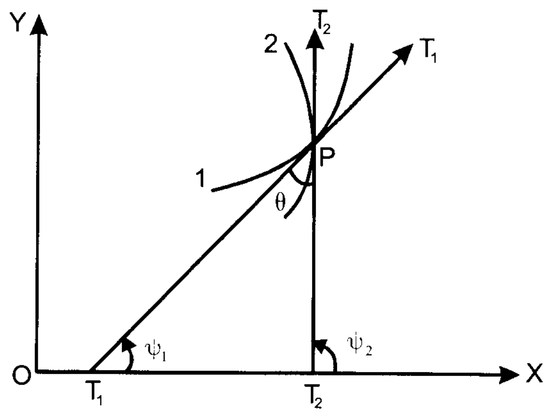
$$x = a(\cos t + t \sin t) \quad y = a(\sin t - t \cos t) \text{ at the point } t.$$

Soulution: Same as above

11.7 Angle of Intersection of Curves

If two curves intersect each other at P, then the angle of intersection of the curves is defined as the angle between the tangents to the curves at P.

Let ψ_1 and ψ_2 be the angles which the tangents to the curves at P make with x-axis



Let $\tan\psi_1 = \left(\frac{dy}{dx}\right)_1$ and $\tan\psi_2 = \left(\frac{dy}{dx}\right)_2$ where $\left(\frac{dy}{dx}\right)_1$ and $\left(\frac{dy}{dx}\right)_2$ mean the value of $\frac{dy}{dx}$ at P to the curves 1 and 2 making angle ψ_1 and ψ_2 respectively with x-axis

If θ is the required angle, then from the figure

$$\theta = \psi_1 \sim \psi_2 = \angle T_1 PT_2$$

$$\Rightarrow \quad \tan\theta = \tan(\psi_1 - \psi_2) = \frac{\tan\psi_1 - \tan\psi_2}{1 + \tan\psi_1 \tan\psi_2}$$

$$\therefore \quad \tan\theta = \frac{\left(\frac{dy}{dx}\right)_1 - \left(\frac{dy}{dx}\right)_2}{1 + \left(\frac{dy}{dx}\right)_1 \left(\frac{dy}{dx}\right)_2}$$

$$\theta = \tan^{-1}\left[\frac{\left(\frac{dy}{dx}\right)_1 - \left(\frac{dy}{dx}\right)_2}{1+\left(\frac{dy}{dx}\right)_1\left(\frac{dy}{dx}\right)_2}\right]$$

If the equation of the curve is f(x, y) = 0 and ϕ(x, y) = 0 then the slopes of the tangents at any point to these curves are respectivelty

$$-\frac{fx}{fy} \text{ and } -\frac{\phi_x}{\phi_y}$$

$$\therefore \quad \tan\theta = \frac{\frac{fx}{fy} \sim \frac{\phi_x}{\phi_y}}{1+\frac{fx}{fy}\frac{\phi_x}{\phi_y}}$$

$$\text{i.e} \quad \theta = \tan^{-1}\frac{f_x\,\phi_y \sim f_y\phi_x}{f_x\,\phi_x + f_y\phi_y}$$

If these two curves touch then $\theta=0$ and $\tan\theta=0$. Hence the condition that the two curves touch each other is $f_x\,\phi_y \sim f_y\phi_x = 0$

$$\text{i.e} \quad \frac{f_x}{\phi_x} = \frac{f_y}{\phi_y} \quad \ldots\ldots\ldots\ldots(i)$$

Again two curves are said to cut orthogonally, if the angle between the tangents at their common point of intersection is $\frac{\pi}{2}$ i.e $\theta=\frac{\pi}{2}$ and $\tan\theta=\infty$

Hence the condition that the two curves cut each other orthogonally is $f_x\phi_x + f_y\phi_y = 0$

If the equation of the curves by y = f(x) and y = ϕ(x), then slopes of the tangents at any point to these curves are respectively f'(x) and ϕ (x)

$$\therefore \tan\theta = \frac{f'(x) \sim \phi'(x)}{1+f'(x)\phi'(x)}$$

Hence if the curves touch each other $\theta=0$ and $\tan\theta=0$ and

$$\therefore f'(x) = \phi'(x)$$

If the curves cut each other orthogonally $\theta = \frac{\pi}{2}$ and $\tan\theta = \infty$ and

$\therefore f'(x).\phi'(x) = -1$

Illustrative Example

Example 1. Find the angle of intersection of the curves $2y^2 = x^3$ and $y^2 = 32x$

Solution: Let the equation of the curves be

$f(x, y) = 2y^2 - x^3 = 0$ and $\phi(x, y) = y^2 - 32x = 0$

Then $f_x = -3x^2$, $f_y = 4y$, $\phi_x = -32$, $\phi_y = 2y$

If θ is the angle between the curves then

$$\tan\theta = \frac{f_x\phi_y - f_y\phi_x}{f_x\phi_x + f_y\phi_y} = \frac{(-3x^2)(2y) - 4y(-32)}{(-3x^2)(-32) + (4y)(2y)}$$

$$\Rightarrow \quad \tan\theta = \frac{-6x^2y + 128y}{96x^2 + 8y^2} \quad \text{............(i)}$$

For the point of intersection of these curves we have

$2y^2 = x^3$ and $y^2 = 32x$

$\Rightarrow \quad 2(32x) = x^3 \Rightarrow 64x = x^3 \Rightarrow x(64 - x^2) = 0 \Rightarrow x = 0, \pm 8$

$\therefore \quad y^2 = 32x \Rightarrow y^2 = 0, \pm 256 \Rightarrow y = 0, \pm 16$

at $(0, 0)$, $\tan\theta = 0 \Rightarrow \theta = 0^0$

$$\text{at } (8, 16) \ \tan\theta = \frac{-6.8^2.16 + 128.16}{96.8^2 + 8.16^2} = \frac{2048 - 6144}{6144 + 2048}$$

$$= -\frac{2048}{8192} = -\frac{1}{4}$$

$\Rightarrow \theta = \tan^{-1}\left(-\frac{1}{4}\right)$

Example 2. Show that the curves $x^3 - 3x^2 + 2 = 0$ and $3x^2y - y^3 = 2$ cut orthogonally.

Solution: Let the equation of the curves be

$f(x, y) = x^3 - 3x^2 + 2 = 0$ and $\phi(x, y) = 3x^2y - y^3 - 2 = 0$

Then $f_x = 3x^2 - 3y^2$, $f_y = -6xy$

$\phi_x = 6xy$, $\phi_y = 3x^2 - 3y^2$

Now $f_x\phi_x + f_y\phi_y = (3x^2 - 3y^2)\, 6xy + (-6xy)(3x^2 - 3y^2)$

$= 0$

Hence the curves cut orthogonally

Example 3. Show that the curves $ax^2 + by^2 = 1$ and $a^1x^2 + b^1y^2 = 1$ intersect orthogonally if $\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}$

Solution: Let the equation of the curves be

$f(x, y) = ax^2 + by^2 - 1 = 0$ and $\phi(x, y) = a^1x^2 + b^1y^2 - 1 = 0$

Then $f_x = 2ax$, $f_y = 2by$, $\phi_x = 2a^1x$, $\phi_y = 2b^1x$

If the curves cut each other orthogonally, then

$f_x\,\phi_x + f_y\,\phi_y = 0$

i.e $\quad 2ax\, 2a^1x + 2by + b^1y^2 = 0$

$\Rightarrow \quad aa^1x^2 + bb^1y^2 = 0$ ....(1)

Also $ax^2 + by^2 = 1$ and $a^1x^2 + b^1y^2 = 1$

$\Rightarrow \quad (a-a^1)x^2 + (b-b^1)y^2 = 0$ ....(2) (by subtracting)

Comparing (1) and (2) we get

$$\frac{a-a^1}{aa^1} = \frac{b-b^1}{bb^1}$$

$$\Rightarrow \quad \frac{1}{a} - \frac{1}{a^1} = \frac{1}{b} - \frac{1}{b^1}$$

$$\Rightarrow \quad \frac{1}{a} - \frac{1}{b} = \frac{1}{a^1} - \frac{1}{b^1}$$

Example 4. Show that the curves $y^2 = 2x$ and $2xy = k$ cut at right angles if $k^2=8$

Solution: Let the equation of the curves be

$f(x, y) = y^2 - 2x = 0$ and $\phi(x, y) = 2xy - k = 0$

We have $y^2 = 2x$ i.e. $y = f(x) = \sqrt{2x}$

and $2xy = k$ i.e. $y = \phi(x) = \dfrac{k}{2x}$

$$\therefore\ f'(x) = \frac{\sqrt{2}}{2\sqrt{x}} \text{ and } \phi'(x) = \frac{-k}{2x^2}$$

If the curves cut at right angles then

$$f'(x)\ \phi'(x) = -1$$

or $$\frac{\sqrt{2}}{2\sqrt{x}}\left(-\frac{k}{2x^2}\right) = -1$$

or $$k = \frac{4x^2\sqrt{x}}{\sqrt{2}} \quad \ldots\ldots\ldots\ldots(1)$$

The point of intersection of these two curves are

$$y^3 = k \Rightarrow y = k^{1/3}\ \therefore\ x = \frac{y^2}{2} = \frac{k^{2/3}}{2}$$

$$\therefore \text{ form (1) } k = \frac{4k^{4/3}}{4}\cdot\frac{k^{1/3}}{\sqrt{2}\sqrt{2}} = \frac{k^{5/3}}{2}$$

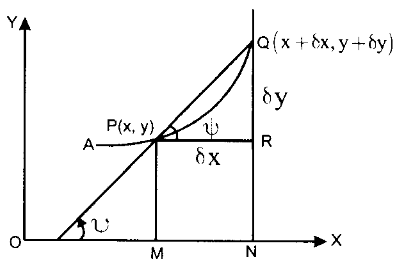
11.8 Differential Coefficient of the length of an are of y = f(x) or x = f(y)

Let S denote the actual distance of any point P(x, y) from some fixed point A of the curve y = f(x). Let Q(x +δx, y +δy) be any other point near P.

Let the are AQ = S +δs

So the are PQ = δs

Now from the figure



$$(\text{chord PQ})^2 = PR^2 + QR^2$$

$$\Rightarrow \left(\frac{\text{chord PQ}}{PR}\right)^2 = 1 + \left(\frac{QR}{PR}\right)^2$$

$$\Rightarrow \left(\frac{\text{chord PQ}}{\text{Arc PQ}}\cdot\frac{\text{Arc PQ}}{PR}\right)^2 = 1 + \left(\frac{QR}{PR}\right)^2$$

$$\Rightarrow \left(\frac{\text{chord PQ}}{\text{Arc PQ}}\right)^2\left(\frac{\text{Arc PQ}}{PR}\right)^2 = 1 + \left(\frac{QR}{PR}\right)^2$$

Now As Q→P along the curve, Arc PQ→Chord PQ and hence

$$\lim_{Q\to P}\frac{\text{Chord PQ.}}{\text{Arc PQ}} = 1$$

Hence in the limiting position as Q→P along the curve PR→0 i.e $\delta x \to 0$ and therefore

$$\lim_{\delta\to 0}\left(\frac{\delta s}{\delta x}\right)^2 = \lim_{\delta x\to 0}\left(1+\left(\frac{\delta y}{\delta x}\right)^2\right)$$

i.e $$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 \text{ ....(i)}$$

or $$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \text{ ....(ii)}$$

Since $\frac{ds}{dy} = \frac{ds}{dx}\cdot\frac{dx}{dy}$ we get on multiplying both sides of (ii) by $\frac{dx}{dy}$ as

$$\frac{ds}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2} \text{ ....(iii)}$$

Again from ΔPQR, $\sin QPR = \frac{QR}{PQ}$

i.e $\sin QPR = \frac{QR}{\text{Arc } PQ}\cdot\frac{\text{Arc } PQ}{PQ}$

as Q→P along the curve, the secant QP becomes the tangent at P and $\angle QPR \to \psi$ where ψ is the angle which the tangent at P makes with x-axis and hence in the limiting position Arc PQ→0 and $\frac{\text{Arc } PQ}{PQ} \to 1$

$$\sin\psi = \lim_{\delta s\to 0}\frac{\delta y}{\delta s} = \frac{dy}{ds} \text{ ....(iv)}$$

Similarly $$\cos\psi = \lim_{\delta s\to 0}\frac{\delta x}{\delta s} = \frac{dx}{ds} \text{ ....(v)}$$

and $$\tan\psi = \frac{dy/ds}{dx/ds} = \frac{dy}{dx} \text{ ....(vi)}$$

Since $\tan\psi = \frac{dy}{dx}$ and $\cot\psi = \frac{dx}{dy}$, we have from (ii) and (iii)

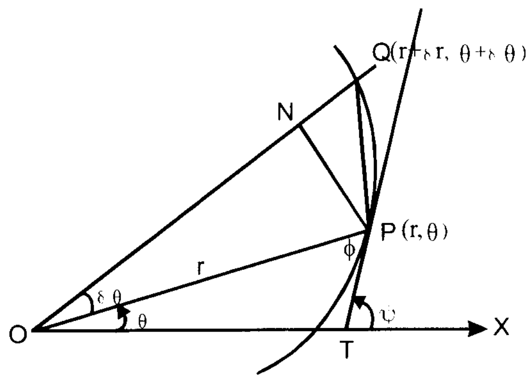
$$\frac{ds}{dx} = \sqrt{1+\tan^2\psi} = \sec\psi \quad ......(vii)$$

and $$\frac{ds}{dy} = \sqrt{1+\cot^2\psi} = \operatorname{cosec}\psi \quad ....(viii)$$

$$\therefore \quad \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 = \frac{1}{\sec^2\psi} + \frac{1}{\operatorname{cosec}^2\psi} = \cos^2\psi + \sin^2\psi = 1$$

11.9 Angle between Radius Vector and Tangent

Let P(r, θ) be a given point on the curve r = f(θ) and Q(r + δr, $\theta + \delta\theta$) be a point on the curve near P



Let QP be the secant through Q and P. PN is drawn perpendicular to OQ.

Then $\angle PON = \delta\theta$

PN = r$\sin\delta\theta$ and ON = r $\cos\delta\theta$

Let ϕ be the angle made by the tangent PT at P with the radius vector OP i.e $\angle OPT = \phi$

From right angled ΔPQN

$$\tan PQN = \frac{PN}{QN} = \frac{PN}{OQ - ON} = \frac{r\sin\delta\theta}{r+\delta r - r\cos\delta\theta}$$

$$\Rightarrow \tan PQN = \frac{r\sin\delta\theta}{r(1-\cos\delta\theta)+\delta r}$$

$$= \frac{r \sin \delta\theta}{r.2\sin^2 \frac{\delta\theta}{2} + \delta r}$$

$$= \frac{r \sin \delta\theta}{\delta\theta} \frac{1}{\frac{1}{2} r\delta\theta \left(\frac{\sin \delta \theta/2}{\delta \theta/2} \right)^2 + \frac{\delta r}{\delta\theta}}$$

(Dividing numberator and denominator by $\delta\theta$)

Now as $Q \to P$ along the curve, $\delta\theta \to 0$ and the secant PQ becomes tangent at P and $\angle PQN \to \angle OPT$ ie $\angle PQN \to \phi$

$$\text{Hence } \tan\phi = \lim_{\delta\theta \to 0} \frac{r\left(\frac{\sin \delta\theta}{\delta\theta}\right)}{\frac{1}{2} r \delta\theta \left(\frac{\sin \delta \theta/2}{\delta \theta/2} \right)^2 + \left(\frac{\delta r}{\delta\theta}\right)}$$

$$\Rightarrow \tan\phi = \frac{r}{\frac{dr}{d\theta}} = r \frac{d\theta}{dr} \text{ ....(ii)}$$

$$[\because \lim_{\delta\theta \to 0} \frac{\sin \delta\theta}{\delta\theta} \text{ and } \lim_{\delta\theta \to 0} \left(\frac{\sin \delta\theta/2}{\delta\theta/2} \right)^2 \text{ are each equal to 1.]}$$

If S denotes the length of the are AP measured from a fixed point A on the curve and S + δs be the length of the are AQ so that arc PQ = δs

$$\text{Then Sin PQN} = \frac{PN}{PQ} = \frac{r \sin \delta\theta}{\delta\theta} \frac{\delta\theta}{\text{arc } PQ} \frac{\text{arc } PQ}{PQ}$$

Now as $Q \to P$, then $\delta\theta \to 0$, $\delta s \to 0$ and $\frac{\text{arc } PQ}{PQ} \to 1$ and $\angle PQN \to \phi$

$$\therefore \text{Sin}\phi = \lim_{\delta s \to 0} r \frac{\delta\theta}{\delta s} = r \frac{d\theta}{dS} \text{ ....(ii)}$$

$$\text{Again Cos PQN} = \frac{QN}{PQ} = \frac{OQ - ON}{PQ} = \frac{(r + \delta r) - r \cos \delta\theta}{PQ}$$

$$\text{i.e } \cos PQN = \frac{r(1-\cos\delta\theta)+\delta r}{PQ}$$

$$= \frac{2.2\sin^2 {}^{\delta\theta}\!/_2 + \delta r}{PQ}$$

$$\text{i.e } \operatorname{Cos} PQN = \frac{1}{2}\, r\delta\theta \left(\frac{\sin {}^{\delta\theta}\!/_2}{{}^{\delta\theta}\!/_2}\right)^2 \cdot \frac{\delta\theta}{\delta s}\cdot\frac{\delta s}{PQ} + \frac{\delta r}{\delta s}\cdot\frac{\delta s}{PQ}$$

Now as $Q \to P$ then as before we get

$$\operatorname{Cos}\phi = \lim_{\delta s\to 0}\frac{\delta r}{\delta s} = \frac{dr}{dS} \quad \text{.............(iii)}$$

Remark: From ΔOPT, $\angle PTX = \angle POT + \angle LOPT$

$\therefore\ \psi = \theta + \phi$

11.10 Derivative of arc length (Polar form)

In the figure of art. 11.8 we have

$$PQ^2 = PN^2 + QN^2 = (r\sin\delta\theta)^2 + (r+\delta r - r\cos\delta\theta)^2$$

$$\Rightarrow\quad PQ^2 = r^2\sin^2\delta\theta + \{r(1-\cos\delta\theta) + \delta r\}^2$$

$$= r^2\sin^2\delta\theta + \{2.r\sin^2 {}^{\delta\theta}\!/_2) + \delta r\}^2$$

Dividing both sides by $\delta\theta^2$ we get

$$\left(\frac{PQ}{\delta\theta}\right)^2 = r^2\left(\frac{\sin\delta\theta}{\delta\theta}\right)^2 + \left\{\frac{1}{2} r\delta\theta\left(\frac{\sin {}^{\delta\theta}\!/_2}{{}^{\delta\theta}\!/_2}\right)^2 + \left(\frac{\delta r}{\delta\theta}\right)^2\right\}$$

$$\Rightarrow\quad \left(\frac{PQ}{\delta s}\cdot\frac{\delta s}{\delta\theta}\right)^2 = r^2\left(\frac{\sin\delta\theta}{\delta\theta}\right)^2 + \left\{\frac{1}{2} r\delta\theta\left(\frac{\sin {}^{\delta\theta}\!/_2}{{}^{\delta\theta}\!/_2}\right)^2 + \left(\frac{\delta r}{\delta\theta}\right)^2\right\}$$

In the limiting position $Q \to P$, $\delta\theta \to O$ and $\dfrac{PQ}{\delta s} \to 1$

$$\therefore\ \lim_{\delta\theta\to 0}\left(\frac{\delta s}{\delta\theta}\right)^2 = \lim_{\delta\theta\to 0} r^2\left(\frac{\sin\delta\theta}{\delta\theta}\right)^2 + \lim_{\delta\theta\to 0}\left\{\frac{1}{2} r\delta\theta\left(\frac{\sin {}^{\delta\theta}\!/_2}{{}^{\delta\theta}\!/_2}\right)^2 + \left(\frac{\delta r}{\delta\theta}\right)^2\right\}$$

$$\Rightarrow \left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 \quad \text{............(i)}$$

$$\text{i.e } \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \quad \text{.............(ii)}$$

Multiplying both sides of (ii) by $\frac{d\theta}{dr}$ we get

$$\frac{ds}{dr} = \sqrt{1 + \left(r\frac{d\theta}{dr}\right)^2} \quad \text{...........(iii)}$$

Cor: Multiplying (i), (ii) and (iii) respectively by $d\theta^2$, $d\theta$, dr we get

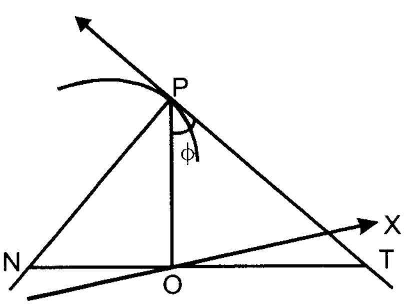
$$ds^2 = dr^2 + r^2\, d\theta^2$$

$$\text{ie} \quad ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, .\, d\theta$$

$$\text{ie} \quad ds = \sqrt{1 + \left(r\frac{d\theta}{dr}\right)^2}\, .\, dr$$

11.12 Subtangent and sub normal (Polar form)

Let NOT be drawn perpendicular to the radius vector OP, through the pole O, meeting the tangent and normal to the curve at T and N respectively. Then OT is called polar sub tangent for the point P and ON is called the polar subnormal for the point P.



Let ϕ be the angle between the radius vector OP and the tangent PT and let OP = r

Now Polar Subtangent = OT = OP $\tan\phi$ (form ΔPON)

$$= r.\ r\frac{d\theta}{dr}\ [\because \tan\phi = r\frac{d\theta}{dr}]$$

$$= r^2\frac{d\theta}{dr}$$

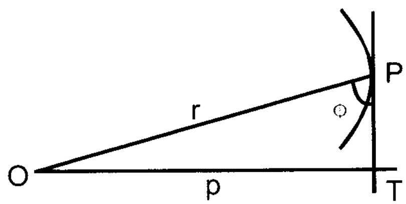
Polar Subnormal = $r\cot\phi = \frac{r}{\tan\phi} = \frac{r}{r\frac{d\theta}{dr}} = \frac{dr}{d\theta}$

11.12. Length of Perpendicular from Pole on the tangent

Let p be the length of the perpendicular OT from the pole O to the tangent PT at the point P of the curve, the length of the radius vector OP being r.

Let ϕ be the angle between the radius vector and the tangent PT.

Now $p = OT = OP\sin\phi = r\sin\phi$



$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi) = \frac{1}{r^2}\left[1+\left(\frac{1}{r}\frac{dr}{d\theta}\right)^2\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \quad ....(i)$$

Putting $u = \frac{1}{r}\ \therefore \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$

$$\therefore \left(\frac{du}{d\theta}\right)^2 = \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

$\therefore$ from (i) $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$ .... (ii)

Note: The relation between p and r for a given curve is called to pedal equation where p is the length of the perpendicular from the pole on the tangent to the curve at any point of it and r is the radius vector of this point.

11.13 To find the pedal equation from the Cartesian equation

If p is the length of the tangent from the origin (0, 0) to the tangent at (x, y)

i.e $\quad Y - y = \frac{dy}{dx}(X - x)$

i.e $\quad X\frac{dy}{dx} - Y + (y - x\frac{dy}{dx}) = 0$ Then

$$p = \frac{y - x\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} \text{ from Coordinate Geometry ......(i)}$$

Also $r^2 = x^2 + y^2$ ....(ii) where r is the radius vector of the point (x, y) and Cartesian equation of the curve is f(x, y) = 0 .....(iii)

Then by elliminating x from (i), (ii) and (iii) we get a relation between p and r which is the require pedal equation of the curve

11.14 To find the pedal equation from polar equations

Let $f(r, \theta) = 0$ be the polar equation of the curve

Then since $\quad f(r, \theta) = 0$ ....(i)

$$\tan\phi = r\frac{d\theta}{dr} \text{ ....(ii)}$$

$$p = r\sin\phi \text{ ....(iii)\\}$$

or $\quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$ .... (iv)

On elliminating θ and ϕ from (i), (ii), (iii), (iv)

We get a relation between p and r which is the required pedal equation.

Illustrative Examples

Example 1. Calculate $\frac{ds}{dx}$ for the parabola $y^2 = 4ax$.

Solution: Here $y^2 = 4ax$

$$\therefore \quad 2y\frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

$$\therefore \quad \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{4a^2}{y^2}}$$

$$\Rightarrow \quad \frac{ds}{dx} = \sqrt{1+\frac{4a^2}{4ax}} = \sqrt{1+\frac{a}{x}} \qquad \because y^2 = 4ax$$

$$\therefore \quad \frac{ds}{dx} = \sqrt{\frac{a+x}{x}}$$

Example 2. In the Cycloid $x = a(1-\cos\theta)$, $y = a(\theta+\sin\theta)$ prove that $\frac{ds}{dx} = \sqrt{\frac{2a}{x}}$

Solution: Here $x = a(1-\cos\theta) \Rightarrow \frac{dx}{d\theta} = a\sin\theta$

$$y = a(\theta+\sin\theta) \Rightarrow \frac{dy}{d\theta} = a(1+\cos\theta)$$

$$\therefore \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a(1+\cos\theta)}{a\sin\theta} = \frac{2\cos^2\theta/2}{2\sin\theta/2\cos\theta/2} = \cot\theta/2$$

$$\text{Now } \frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\cot^2\theta/2} = \operatorname{cosec}\theta/2$$

$$\Rightarrow \quad \frac{ds}{dx} = \frac{1}{\sin\theta/2} = \frac{1}{\sqrt{\frac{2\sin^2\theta/2}{2}}} = \frac{1}{\sqrt{\frac{1-\cos\theta}{2}}}$$

$$\Rightarrow \quad \frac{ds}{dx} = \frac{1}{\sqrt{\frac{x/a}{2}}} \qquad \because x = a(1-\cos\theta)$$

$$\therefore \frac{ds}{dx} = \sqrt{\frac{2a}{x}}$$

Example 3. For the ellipse x = acost, y = bsint, prove that

$$\frac{ds}{dt} = a\left(1-e^2\cos^2 t\right)^{1/2}$$

Solution: Here x = a cos $\Rightarrow \frac{dx}{dt} = -$ a sint

$$y = b\ \text{sint} \Rightarrow \frac{dy}{dt} = b\ \text{cost}$$

$$\therefore \quad \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{a^2\sin^2 t + b^2\cos^2 t}$$

$$\Rightarrow \quad \frac{ds}{dt} = \sqrt{a^2\sin^2 t + b^2(1-e^2)\cos^2 t} \qquad [\because b^2 = a^2(1-e^2)]$$

$$\Rightarrow \quad \frac{ds}{dt} = \sqrt{a^2(\sin^2 t + \cos^2 t) - a^2e^2\cos^2 t}$$

$$\Rightarrow \quad \frac{ds}{dt} = a\sqrt{1-e^2\cos^2 t}$$

Example 4. Calculate $\frac{ds}{dx}, \frac{ds}{dy}$ for the curve y = a log sec $\frac{x}{a}$ and prove that x = aψ. Further prove that $\frac{d^2x}{ds^2} = -\frac{1}{2a}\sin\left(\frac{2x}{a}\right)$

Solution: Equation of the curve is y = a log sec $\frac{x}{a}$

$$\therefore \frac{dy}{dx} = a\frac{1}{\sec x/a}\ \sec\frac{x}{a}\ \tan\frac{x}{a}\,\frac{1}{a}$$

$$\Rightarrow \frac{dy}{dx} = \tan\frac{x}{a} \quad ....(i)$$

$$\text{Now } \frac{ds}{dy} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\tan^2\frac{x}{a}} = \sec\frac{x}{a} \quad ......(ii)$$

$$\frac{ds}{dy} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\cot^2\frac{x}{a}} = \text{cosec } \frac{x}{a} \quad ......(iii)$$

$$\text{Also from (i) } \tan\psi = \tan\frac{x}{a} \qquad \left[\because \tan\psi = \frac{dy}{dx}\right]$$

$$\therefore \quad \psi = \frac{x}{a}$$

$$\text{or} \quad x = a\psi$$

$$\text{Again from (ii) } \frac{dx}{ds} = \frac{1}{\sec\frac{x}{a}} = \cos\frac{x}{a}$$

$$\therefore \frac{d^2x}{ds^2} = -\sin\frac{x}{a}\cdot\frac{1}{a}\frac{dx}{ds}$$

$$= -\sin\frac{x}{a}\cdot\frac{1}{a}\cos\frac{x}{a}$$

$$= -\frac{1}{2a}.2\sin\frac{x}{a}\cos\frac{x}{a}$$

$$= -\frac{1}{2a}\sin\left(\frac{2x}{a}\right)$$

Example 5. Find $\frac{ds}{d\theta}$ for the curves

(i) $r = ae^{\theta\cot\alpha}$ (ii) $r^n = a^n \cos n\theta$

Solution: (i) $r = ae^{\theta\cot\alpha}$

$$\therefore \frac{dr}{d\theta} = a\cot\alpha\, e^{\theta\cot\alpha} = \cot\alpha\left(ae^{\theta\cot\alpha}\right) = r\cot\alpha$$

$$\therefore \left(\frac{ds}{d\theta}\right)^2 = r^2 + \left(\frac{dr}{d\theta}\right)^2 = r^2 + r^2 \cot^2\alpha = r^2 (1 + \cot^2\alpha)$$

$$= r^2 \operatorname{cosec}^2\alpha$$

$$\therefore \frac{ds}{d\theta} = r \operatorname{cosec}\alpha$$

(ii) $r^n = a^n \cos n\theta$

Taking logarithm of both sides we get

$n \log r = n \log a + \log \cos n\theta$ $[\because \log a^m = m \log a, \log ab = \log a + \log b]$

Differentiating both sides we get

$$\frac{n}{r}\frac{dr}{d\theta} = -\frac{n \sin n\theta}{\cos n\theta}$$

$$\Rightarrow \frac{dr}{d\theta} = -r \tan n\theta$$

$$\text{Since } \tan\phi = r\frac{d\theta}{dr} = r.\left(\frac{1}{-t \tan n\theta}\right) = -\cot n\theta$$

$$\therefore \quad \tan\phi = \tan\left(\frac{\pi}{2} + n\theta\right) \Rightarrow \phi = \frac{\pi}{2} + n\theta$$

$$\text{Now } r\frac{d\theta}{ds} = \sin\phi = \sin\left(\frac{\pi}{2} + n\theta\right)$$

$$\therefore \quad r\frac{d\theta}{ds} = \cos n\theta$$

$$\therefore \quad \frac{d\theta}{ds} = \frac{\cos n\theta}{r}$$

$$\therefore \quad \frac{ds}{d\theta} = \frac{r}{\cos n\theta} = r \sec n\theta$$

$$\Rightarrow \quad \frac{ds}{d\theta} = a\,(\cos n\theta)^{1/n}(\cos n\theta)^{-1} \qquad \left[\because \cos\theta = \left(\frac{r}{a}\right)^n\right]$$

$$\Rightarrow \quad \frac{ds}{d\theta} = a\,(\cos n\theta)^{1/n - 1} = a\,(\cos n\theta)^{(1-n)/n}$$

$$\therefore \quad \frac{ds}{d\theta} = a\,(\cos n\theta)^{(1-n)/n}$$

Example 6. Show that in the curve $r = ae^{\theta\cot\alpha}$

(i) polar sub tangent = r $\tan\alpha$

(i) polar sub normal = r $\cot\alpha$

Solution: Here $r = ae^{\theta\cot\alpha}$

$$\therefore \quad \frac{dr}{d\theta} = a\cot\alpha e^{\theta\cot\alpha} = \cot\alpha(ae^{\theta\cot\alpha})$$

$$\Rightarrow \quad \frac{dr}{d\theta} = r\cot\alpha$$

$$\therefore \quad \text{polar subnormal} = \frac{dr}{d\theta} = r\cot\alpha$$

$$\text{Again } \frac{d\theta}{dr} = \frac{1}{r\cot\alpha} = \frac{1}{r}\tan\alpha$$

$$\therefore \text{ polar subtangent} = r^2\frac{d\theta}{dr} = r^2\left(\frac{1}{r}\tan\alpha\right) = r\tan\alpha$$

Example 7. In the Cardiode $r = a(1 - \cos\theta)$, prove that the polar subtangent is $\dfrac{2a\sin^3 \theta/2}{\cos \theta/2}$

Solution: Here $r = a(1-\cos\theta)$

$$\therefore \quad \frac{dr}{d\theta} = a\sin\theta$$

$$\therefore \quad \frac{d\theta}{dr} = \frac{1}{a\sin\theta}$$

$$\therefore \quad r^2\frac{d\theta}{dr} = r^2\frac{1}{a\sin\theta} = \frac{a^2(1-\cos\theta)^2}{a\sin\theta}$$

$$\Rightarrow \quad r^2\frac{d\theta}{dr} = \frac{a\left(2\sin^2 \theta/2\right)^2}{2\sin\theta/2\cos\theta/2} = \frac{4a\sin^4 \theta/2}{2\sin\theta/2\cos\theta/2}$$

$$\therefore \quad r^2 \frac{d\theta}{dr} = \frac{2a\sin^3 \theta/2}{\cos \theta/2}$$

Hence polar sub tangent $= r^2 . \frac{d\theta}{dr} = \frac{2a\sin^3 \theta/2}{\cos \theta/2}$

Example 8. Find the pedal equation ofthe parabola $y^2 = 4ax$ with respect to its vertex.

Solution: Here the curve is $y^2 = 4ax$ ....(i)

$$\therefore \quad 2y \frac{dy}{dx} = 4a$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y}$$

The equation of the tangent to the curve (i) at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

$$\text{or} \quad Y - y = \frac{2a}{y}(X - x)$$

or $\quad Yy - y^2 = 2aX - 2ax$

or $\quad Yy - 4ax = 2aX - 2ax \qquad \because y^2 = 4ax$

or $\quad 2aX - Yy + 2ax = 0$ ....(ii)

If p is the length of the perpendicular to the tangent (ii) from the origin (0, 0) then

$$p = \frac{2ax}{\sqrt{(2a)^2 + (-y)^2}} = \frac{2ax}{\sqrt{4a^2 + y^2}}$$

$$\text{or} \quad p = \frac{2ax}{\sqrt{4a^2 + 4ax}} = \frac{x\sqrt{a}}{\sqrt{a + x}} \quad \text{.......(iii)}$$

Also $r^2 = x^2 + y^2 = x^2 + 4ax$ ....(iv)

from (iii) we get $p^2 (a + x) = ax^2$ (by squaring both sides)

or $\quad ax^2 - p^2x - ap^2 = 0$

or $x = \frac{1}{2a}\left[p^2 + \sqrt{p^4 + 4a^2p^2}\right]$

putting the value of x in (iv) we get

$$r^2 = \frac{1}{4a2}\left[p^2 + \sqrt{p^4 + 4a^2p^2}\right]^2 + 4a.\frac{1}{2a}\left[p^2 + \sqrt{p^4 + 4a^2p^2}\right]$$

or $4a^2r^2 = p^4 + (p^4 + 4a^2p^2) + 2p^2\sqrt{p^4 + 4a^2p^2}$

$$+ 8a^2p^2 + 8a^2\sqrt{p^4 + 4a^2p^2}$$

$\Rightarrow$ $4a^2r^2 - 2p^4 - 12a^2p^2 = \sqrt{p^4 + 4a^2p^2}\,(8a^2 + 2p^2)$

or $(4a^2r^2 - 2p^4 - 12a^2p^2)^2 = (8a^2 + 2p^2)^2(p^4 + 4a^2p^2)$

which is the required pedal equation.

Example 9. Find the pedal equation of the circle $x^2 + y^2 = 2ax$

Solution: The circle is $x^2 + y^2 = 2ax$ ....(i)

Differentiating both sides w.r.t x

$$2x + 2y = \frac{dy}{dx} = 2a$$

or $\frac{dy}{dx} = \frac{a - x}{y}$

The equation of the tangent to the circle (i) at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x)$$

or $Y - y = \frac{a - x}{y}(X - x)$

or $(x - a)X + yX = -ax + x^2 + y^2$

or $(x - a)X + yY - ax = 0$ ....(ii) $\because x^2 + y^2 = 2ax$

It p is the length of the perpendicular on the tangent (ii) from the origin (0, 0) then

$$p = \frac{ax}{\sqrt{(x - a)^2 + y^2}} = \frac{ax}{\sqrt{x^2 + a^2 - 2ax + y^2}}$$

or $p = \frac{ax}{\sqrt{x^2 + y^2 - 2ax + a^2}} = \frac{ax}{\sqrt{a^2}} = x$ ......(iii)

Also $r^2 = x^2 + y^2 = 2ax$ from (i)

$\therefore$ $r^2 = 2ap$ is the required pedal equation.

Exercises

1. Find the equation of the tangent at the point (x, y) of the following curves:
   (i) $x^3 + y^3 - 3axy = 0$ (ii) $(x^2 + y^2)^2 = a^2 (x^2 + y^2)$

   (iii) $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 0$ (iv) $x^{2/3} + y^{2/3} = a^{2/3}$

2. Find the equation of the tangent at the point 'θ' on each of the following curves:
   (i) $x = a\cos 2\theta;\ y = 2\sqrt{2}\,a\sin\theta$
   (ii) $x = a\cos\theta;\ y = b\sin\theta$
   (iii) $x = a\cos\theta;\ y = a\sin\theta$

3. Find the equation of the tangent to the curve
   (i) $x = at^2;\ y = at^3$ at any point t
   (ii) $x = at^2;\ y = 2at$ at any point t

4. Find the normal to the curves $\sqrt{xy} = a+x$ which makes equal intercepts an co-ordinate axes

5. Find at what points to the curve $y = (x-3)^2 (x-2)$ the tangent is parallel to the x-axis.

6. Find where the tangent is parallel to x-axis for the curves

   (i) $\frac{x^3}{a} + \frac{y^3}{b} = xy$ (ii) $y = x^3 - 3x^2 - 9x + 15$

7. Find where the tangent is parallel to y-axis for the curves
   (i) $y^2 = x^2(a-x)$ (ii) $y = (x-3)^2 (x-2)$

8. Find the equation of the tangent and that of the normal at any point of the ellipse $a^2y^2 + b^2x^2 = a^2b^2$

9. Prove that $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $\frac{x}{a} + \log\left(\frac{y}{b}\right) = 0$

10. Prove that all points on the curve $y^2 = 4a\{x + a \sin\left(\frac{x}{a}\right)\}$ at which the tangent is parallel to the x-axis lie on a parabola.

11. If $p = x\cos\alpha + y\sin\alpha$ touch the curve $\left(\frac{x}{a}\right)^{n/(n-1)} + \left(\frac{y}{b}\right)^{n/(n-1)} = 1$. Prove that $p^n = (a\cos\alpha)^n + (b\sin\alpha)^n$

12. Prove that the line $\frac{x}{a} + \frac{y}{b} = 1$ touches the curve $y = be^{-x/a}$ at the point where the curve crosses the y-axis.

13. If the line $lx+my = 1$ touches the curve $(ax)^n + (by)^n = 1$, the prove that $\left(\frac{l}{a}\right)^{n/(n-1)} + \left(\frac{m}{b}\right)^{n/(n-1)} = 1$

14. If the line $x\cos\alpha + y\sin\alpha = p$ touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, show that $p^2 = (a\cos\alpha)^2 + (b\sin\alpha)^2$

15. Find the length of the sub tangent sub normal lengths of tangent and normal to the following curves

    (i) $x = a(\theta+\sin\theta)$; $y = a(1-\cos\theta)$ at the point θ

    (ii) $x = a(\cos t+t\sin t)$; $y = a(\sin t-t\cos t)$ at the point t

16. Prove that the sub tangent at any point on the curve $x^m y^m = a^{m+n}$ varies as the abscissa of the point.

17. Prove that in the curve $y = be^{x/a}$ the sub tangent is constant and sub normal is $y^{2/a}$

18. Show that in any curve $\frac{\text{sub normal}}{\text{sub tangent}} = \left[\frac{\text{length of normal}}{\text{length of tangent}}\right]^2$

19. Find the angle of intersection of the following curves

    (i) $xy = a^2$ and $x^2 + y^2 = 2a^2$ (ii) $2y^2 = x^3$ and $y^2 = 32x$

    (iii) $x^2 = 4y$ and $y(x+4) = 8$ (ii) $x^2 - y^2 = 2a^2$ and $x^2 + y^2 = 4a^2$

20. Show that the curve $x^3 - 3xy^2 + 2 = 0$ and $3x^3y - y^2 = 2$ cut orthogonally

21. Find $ds/d\theta$ for the following curves:

    (i) $r^2 = a^2\cos2\theta$ (ii) $r = a(1+\cos\theta)$

22. Show that for the curve $r = e^{\theta}$, the polar sub tangent is equal to the sub-normal

23. Prove that for the curve $r\theta=\alpha$, the polar sun tangent is constant.

24. Prove that for the curve $r = a\theta$, the polar subnormal is constant.

25. For the Cordiode $r = a(1-\cos\theta)$ prove that

    (i) $\phi = \frac{1}{2}\theta$ (ii) $2ap^2 = r^3$

    (iii) polar sub tangent $= 2a\sin^2\frac{\theta}{2}\tan\frac{\theta}{2}$

26. Find the pedal equation of the following curves:

    (i) $r = ae^{\theta\cot\alpha}$ (ii) $r^2\cos2\theta = a^2$ (iii) $r^2 = a^2\cos2\theta$

    (iv) $\frac{1}{r} = e\cos\theta$

27. Show that the portion of the tangent at any point on the curve $x=a\cos^3\theta$; $y=a\sin^2\theta$ intercepted between the axes is of constant length.

28. Show that the pedal equation of

    (i) the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with regard to the centre is $a^2b^2/p^2 = a^2 + b^2 - r^2$

    (ii) the parabola $y^2 = 4a(x+a)$ is $p^2 = ar$

    (iii) the astroid $x^{2/3}+y^{2/3}=a^{2/3}$ is $r^2 + 3p^2 = a^2$

12

Curvature

Introduction

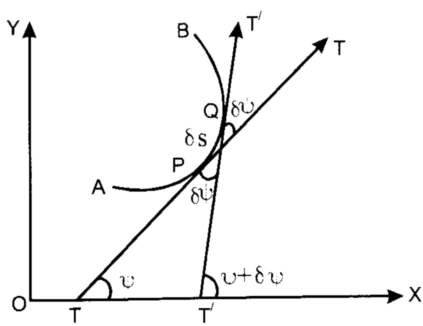
The terms flatness or sharpness are often used to describe the nature of bending of the curve at a particular point. In this chapter we give the mathematical expression of curvature of the curve at a particular point which give a definite numerical measure of bending which the curve undergoes at the point.

We shall assume that whenever dirivatives occur in our problems they do exists at the points under consideration.

12.1 Definintions

(A) Angle of Contingency

Let P and Q be two neighbouring points on a curve and the tangentS PT and QT$'$ at these two points make angles ψ and $\psi+\delta\psi$ respectively with a fixed



line the x-axis. Let s be the length of the arc AP measured from the fined point A and let AQ be $s+\delta s$ such that $PQ = \delta s$.

Here $\angle PTX = \psi$, $\angle QT'X = \psi+\delta\psi$, hence

$$\angle TPT' = (\psi+\delta\psi) - \psi = \delta\psi$$

Thus $\delta\psi$ measures the change in the inclination of the tangent line as the point of contanct P moves along the curve to describe $PQ = \delta s$. The angle $\delta\psi$ is called the angle of contingency of the arc PQ which is the difference of the angles which the tangents at the extremites of the arc make with a fixed straight line.

(B) Average Curvature

The fraction $\dfrac{\delta\psi}{\delta s} = \dfrac{\text{angle of contingency}}{\text{length of arc PQ}}$ is called the average curvature or the average bending of the arc PQ

(C) Curvature

Curvature at any point P of the curve is called the limiting value if it exists of the average curvature as the point Q approaches P along the curve.

$$\text{Thus Curvature at P} = \lim_{Q\to P} \frac{\text{angle of contingency}}{\text{length of arc PQ}}$$

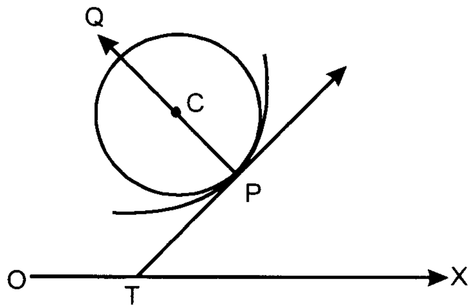
$$= \lim_{\delta s\to 0} \frac{\delta\psi}{\delta s} = \frac{d\psi}{ds}$$

(D) Radius of Curvature

The reciprocal of the curvature at any point P is called the radius of curvature and is denoted by the Greek letter ρ

$$\text{Radius of Curvature } \rho = \frac{ds}{d\psi}$$

If the length $PC = \rho$ measured from P along the positive direction of the normal, the point C is called the centre of curvature at P and the circle with centre C and radius $CP = \rho$ is called circle of curvature at P. Any chord of this circle through the point of contact is called a chord of curvature.



12.2 Formula for Radius of Curvature

(A) Radius of Curvature of the curve y = f(x)

We have $\tan\psi = \frac{dy}{dx}$

Differentiating both sides w.r.t x, we get

$$\sec^2\psi \frac{d\psi}{dx} = \frac{d^2y}{dx^2}$$

or $$\sec^2\psi \frac{d\psi}{ds}\cdot\frac{ds}{dx} = \frac{d^2y}{dx^2}$$

or $$\sec^2\psi \frac{1}{\rho}\sec\psi = \frac{d^2y}{dx^2} \qquad \left[\because \cos\psi = \frac{dx}{ds}\right]$$

or $$\rho = \frac{\sec^3\psi}{\frac{d^2y}{dx^2}} = \frac{\left(1+\tan^2\psi\right)^{3/2}}{\frac{d^2y}{dx^2}}$$

or $$\rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} \qquad \left[\because \tan\psi = \frac{dy}{dx}\right]$$

or $$\rho = \frac{\left(1+y_1^2\right)^{3/2}}{y_2} \text{ where } y_1 = \frac{dy}{dx} \text{ and } y_2 = \frac{d^2y}{dx^2}$$

Cor: Curvature $= \frac{1}{\rho} = \frac{d^2y}{dx^2}\cos^3\psi$

(B) Radius of Curvature of the Curve f(x, y) = 0

$$f(x, y) = 0 \quad \therefore \quad fx + fy\frac{dy}{dx} = 0 \text{ or } \frac{dy}{dx} = -\frac{f_x}{f_y}$$

where f_x, f_y represents the partial differential coefficients of f(x, y) w.r.t x and y respectively

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(-\frac{f_x}{f_y}\right)$$

$$\text{or} \quad \frac{d^2y}{dx^2} = -\frac{1}{(f_y)^2}\left[f_y\left(f_{xx} + f_{xy}\frac{dy}{dx}\right) - f_x\left(f_{xy} + f_{xy}\frac{dy}{dx}\right)\right]$$

Replacing $\frac{dy}{dx}$ by $-\frac{fx}{fy}$ we get

$$\frac{d^2y}{dx^2} = \frac{1}{(f_y)^3}\left[f_{xx}(f_y)^2 - f_{xy}f_xf_y + f_{yy}(f_x)^2\right]$$

$$\therefore \quad \rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left\{1 + \left(-\frac{fx}{fy}\right)^2\right\}^{3/2}}{-\frac{1}{(f_y)^3\left[f_{xx}(f_y)^2 - f_{xy}f_xf_y + f_{yy}(f_x)^2\right]}}$$

$$\text{or} \quad \rho = \frac{\left[(f_x)^2 + (f_y)^2\right]^{3/2}}{f_{xx}(f_y)^2 - 2f_{xy}f_xf_y + f_{yy}(f_x)^2}$$

(C) Radius of Curvature of the Curve x = f(t), y = ϕ(t)

$$\text{Since} \quad \frac{dy}{dx} = \frac{\phi'(t)}{f'(t)} = \frac{y'}{x'}$$

$$\text{and} \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y'}{x'}\right) = \left(\frac{d}{dt}\frac{y'}{x'}\right)\frac{dt}{dx}$$

$$\text{or} \quad \frac{d^2y}{dx^2} = \frac{x'y'' - y'x''}{x'^2} \cdot \frac{1}{x'} = \frac{x'y'' - y'x''}{x'^3}$$

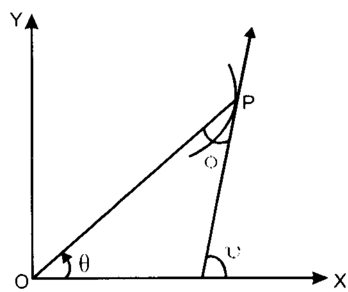
$$\therefore \quad \rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left\{1 + \left(\frac{y'}{x'}\right)^2\right\}^{3/2}}{\frac{x'y'' - y'x''}{x'^3}}$$

$$\text{or} \quad \rho = \frac{\left(x'^2 + y'^2\right)^{3/2}}{x'y'' - y'x''}$$

(D) Formula for Pedal Equation

The angles θ, ϕ, ψ are connected by the relation

$\psi = \theta + \phi$ ....(i)



Differentiating (i) w.r.t s we get

$$\frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\phi}{ds}$$

$$\text{or} \quad \frac{1}{\rho} = \frac{d\theta}{ds} + \frac{d\phi}{dr} \cdot \frac{dr}{ds}$$

$$\text{or} \quad \frac{1}{\rho} = \frac{1}{r}\sin\phi + \cos\phi \frac{d\phi}{dr} \quad \left[\frac{d\theta}{ds} = r\sin\phi \text{ and } \frac{dr}{ds} = \cos\phi\right]$$

$$\text{or} \quad \frac{1}{\rho} = \frac{1}{r}\left[\sin\phi + r\cos\phi \frac{d\phi}{dr}\right]$$

$$\Rightarrow \quad \frac{1}{\rho} = \frac{1}{r} \cdot \frac{d}{dr}(r\sin\phi) = \frac{1}{r}\frac{dp}{dr} \qquad [\because p = r\sin\phi]$$

$$\therefore \quad \rho = r\frac{dr}{dp}$$

(E) Formula for Polar Equations

We know that $\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$ ....(i)

Differfentiating (i) w.r.t r we get

$$-\frac{2}{p^3}\frac{dp}{dr}=\frac{-2}{r^3}-\frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2-\frac{1}{r^4}\frac{d^2r}{d\theta^2}\,2\frac{dr}{d\theta}.\frac{d\theta}{dr}$$

or $$\frac{1}{p^3}\frac{dp}{dr}=\frac{1}{r^3}+\frac{2}{r^5}\left(\frac{dr}{d\theta}\right)^2-\frac{1}{r^4}\frac{d^2r}{d\theta^2}$$

$$\Rightarrow\quad \frac{1}{p^3}\frac{1}{r}\frac{dp}{dr}=\frac{1}{r^4}+\frac{2}{r^6}\left(\frac{dr}{d\theta}\right)^2-\frac{1}{r^5}\frac{d^2r}{d\theta^2}$$

$$\Rightarrow\quad \frac{1}{r}\frac{dp}{dr}=p^3\left[\frac{1}{r^4}+\frac{2}{r^6}\left(\frac{dr}{d\theta}\right)^2-\frac{1}{r^5}\frac{d^2r}{d\theta^2}\right]$$

$$\Rightarrow\quad \frac{1}{\rho}=\frac{1}{r}\frac{dp}{dr}=p^3\frac{1}{\left\{\frac{1}{r^2}+\frac{2}{r^4}\left(\frac{dr}{d\theta}\right)^2\right\}^{3/2}}\left[\frac{1}{r^4}+\frac{2}{r^6}\left(\frac{dr}{d\theta}\right)^2-\frac{1}{r^5}\frac{d^2r}{d\theta^2}\right]\text{ by (i)}$$

$$\Rightarrow\quad \frac{1}{p}=\frac{1}{\left(\frac{1}{r^4}\right)^{3/2}\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}\left[\frac{r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}}{r^6}\right]$$

$$\Rightarrow\quad \frac{1}{\rho}=\frac{r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}}{\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}$$

$$\therefore\quad \rho=\frac{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}}$$

(F) Polar Tangental Formula

To find the radius of curvature of the curve for which the relation between p and ψ is given

We know that $\frac{dr}{ds} = \cos\varphi$ and $\rho = \frac{ds}{d\psi} = r\frac{dr}{dp}$

Now $\frac{dp}{d\psi} = \frac{dp}{dr}.\frac{dr}{ds}.\frac{ds}{d\psi} = \frac{dp}{dr}\cos\varphi\left(r\frac{dr}{dp}\right)$

or $\frac{dp}{d\psi} = r\cos\varphi$ ....(i)

Also $p = r\sin\phi$ ....(ii)

Squaring and adding (i) and (ii) we get

$p^2 + \left(\frac{dp}{d\psi}\right)^2 = r^2$ ....(iii)

Differentiating (iii) w.r.t p we get

$$2p + 2\frac{dp}{d\psi}.\frac{d^2p}{d\psi^2}.\frac{d\psi}{dp} = 2r\frac{dr}{dp}$$

$$\Rightarrow \quad p + \frac{d^2p}{d\psi^2} = r\frac{dr}{dp}$$

$$\Rightarrow \quad \rho = p + \frac{d^2p}{d\psi^2} \qquad \left[\because \rho = r\frac{dr}{dp}\right]$$

Illustrative Examples

Example 1. Find the radius of curvature at the point (x, y) on the parabola $y^2 = 4ax$

Solution: The curve is $y^2 = 4ax$ ....(i)

$$\therefore \quad 2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2a}{y}$$

and $$\frac{d^2y}{dx^2} = -\frac{2a}{y^2}\frac{dy}{dx} = -\frac{2a}{y^2}\frac{2a}{y}$$

ie $$\frac{d^2y}{dx^2} = -\frac{4a^2}{y^3}$$

$$\therefore \quad \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left[1+\left(\frac{2a}{y}\right)^2\right]^{\frac{3}{2}}}{-\frac{4a^2}{y^3}}$$

$$\Rightarrow \quad \rho = \frac{\left(y^2+4a^2\right)^{\frac{3}{2}}}{4a^2} = \frac{\left(4ax+4a^2\right)^{\frac{3}{2}}}{4a^2}$$

or $$\rho = \frac{(4a)^{\frac{3}{2}}(x+a)^{\frac{3}{2}}}{4a^2} = \frac{2}{\sqrt{a}}(x+a)^{\frac{3}{2}}$$

Example 2. Find the radius of curvature at (x, y) on the curve $a^2y=x^3-a^3$

Solution: The given curve is $a^2y = x^3 - a^3$

$$\Rightarrow \quad a^2\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{3x^2}{a^2}$$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{6x}{a^2}$$

$$\therefore \quad \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left[1+\left(\frac{3x^2}{a^2}\right)^2\right]^{\frac{3}{2}}}{\frac{6x}{a^2}}$$

$$\therefore \quad \rho = \frac{\left(a^4+9x^4\right)^{\frac{3}{2}}}{6xa^4}$$

Example 3. Find the radius of curvature at the point (x, y) on the curve $x^{2/3} + y^{2/3} = a^{2/3}$

Solution: The curve is $x^{2/3} + y^{2/3} = a^{2/3}$

Differentiating both sides w.r.t x we get

$$\frac{2}{3}x^{-1/3} + \frac{2}{3}y^{-1/3}\frac{dy}{dx} = 0$$

or $$\frac{dy}{dx} = -\left(\frac{x}{y}\right)^{-\frac{1}{3}} = -\left(\frac{y}{x}\right)^{\frac{1}{3}}$$

Differentiating again w.r.t x we get

$$\frac{d^2y}{dx^2} = -\frac{x^{1/3}\frac{1}{3}y^{-2/3}\frac{dy}{dx} - y^{1/3}\frac{1}{3}x^{-2/3}}{x^{2/3}}$$

or $$\frac{d^2y}{dx^2} = -\frac{1}{3}\left[\frac{x^{1/3}y^{-2/3}\left(-\frac{y}{x}\right)^{\frac{1}{3}} - y^{1/3}x^{-2/3}}{x^{2/3}}\right]$$

or $$\frac{d^2y}{dx^2} = \frac{1}{3}\frac{\left(x^{2/3} + y^{2/3}\right)}{x^{4/3}y^{1/3}}$$

or $$\frac{d^2y}{dx^2} = \frac{1}{3}\frac{a^{2/3}}{x^{4/3}y^{1/3}} \qquad \left[x^{2/3} + y^{2/3} = a^{2/3}\right]$$

$$\therefore \quad \rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left[1 + \left(\frac{y}{x}\right)^{2/3}\right]^{3/2}}{\frac{1}{3}\frac{a^{2/3}}{x^{4/3}y^{1/3}}}$$

$$\Rightarrow \quad \rho = 3\,\frac{\frac{1}{x}\left[x^{2/3}+y^{2/3}\right]^{3/2}}{a^{2/3}}\,.\,x^{4/3}\,y^{1/3}$$

$$\Rightarrow \quad \rho = 3\,\frac{\left(a^{2/3}\right)^{3/2}}{a^{2/3}}\,.\,x^{4/3}\,y^{1/3} = 3a^{1/3}\,x^{1/3}y^{1/3}$$

Example 4. Show that the radius of curvature at $(a\cos^3\theta,\ a\sin^3\theta)$ on the curve $x^{2/3}+y^{2/3}=a^{2/3}$ is $3a\sin\theta\cos\theta$

Solution: Using the above result and putting

$x = a\cos^3\theta,\ y = a\sin^3\theta$ we get

$$\rho = 3a^{1/3}\ a^{1/3}\cos\theta\,.\,a^{1/3}\sin\theta = 3a\sin\theta\cos\theta$$

Example 5. Find the radius of curvature on the curve $xy = c^2$ at the point (x, y)

Solution: The curve is $xy = c^2$ or $y = \dfrac{c^2}{x}$

$$\therefore \quad \frac{dy}{dx} = -\frac{c^2}{x} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{2c^2}{x^3}$$

$$\therefore \quad \rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left(1+\frac{c^4}{x^4}\right)^{\frac{3}{2}}}{\frac{2c^2}{x^3}} = \frac{\left(x^4+c^4\right)^{\frac{3}{2}}.x^3}{2c^2.x^6}$$

$$\text{or} \quad \rho = \frac{\left(x^4+x^2y^2\right)^{\frac{3}{2}}.x^3}{2c^2x^3} = \frac{\left(x^2+y^2\right)^{\frac{3}{2}}}{2c^2}$$

Example 6. Find the radius of curvature at (x, y) of the curve $y = c\log\sec\left(\dfrac{x}{c}\right)$

Solution: The curve is $y = c\log\sec\left(\dfrac{x}{c}\right)$

$$\therefore \quad \frac{dy}{dx} = c\,\frac{1}{\sec\left(\frac{x}{c}\right)}\sec\left(\frac{x}{c}\right)\tan\left(\frac{x}{c}\right)\frac{1}{c}$$

or $\quad \dfrac{dy}{dx} = \tan\left(\dfrac{x}{c}\right)$

$\therefore \quad \dfrac{d^2y}{dx^2} = \dfrac{1}{c}\sec^2\left(\dfrac{x}{c}\right)$

$$\therefore \quad \rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left\{1+\tan^2\left(\frac{x}{c}\right)\right\}^{\frac{3}{2}}}{\frac{1}{c}\sec^2\left(\frac{x}{c}\right)}$$

$$\text{or} \quad \rho = c\,\frac{\left(\sec^2\left(\frac{x}{c}\right)\right)^{\frac{3}{2}}}{\sec^2\left(\frac{x}{c}\right)} = c\sec\left(\frac{x}{c}\right)$$

Example 7. Prove that for the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ $\rho = \dfrac{a^2b^2}{p^2}$, p being the perpendicular from the centre on the tangent at (x, y)

Solution: Equation of the tangent at (x, y) to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is

$$\frac{Xx}{a^2}+\frac{Yy^2}{b^2}=1$$

If p is the perpendicular from the centre (0, 0) on this tangent then p =

$$\frac{1}{\sqrt{\left(\frac{x}{a^2}\right)^2+\left(\frac{y}{b^2}\right)^2}} = \frac{a^2b^2}{\sqrt{b^4x^2+a^4y^2}} \quad \text{..........(ii)}$$

Given ellipse is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ or $y^2 = \dfrac{b^2}{a^2}(a^2 - x^2)$

$$\therefore \quad 2y\,\frac{dy}{dx} = -\frac{2xb^2}{a^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{b^2}{a^2}\,\frac{x}{y}$$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\left[\frac{y + x\frac{dy}{dx}}{y^2}\right]$$

$$= \frac{b^2}{a^2y^2}\left[\frac{x^2b^2}{ya^2} + y\right]$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = -\frac{b^2}{a^2}\,\frac{x^2b^2 + a^2y^2}{y^3a^2} = -\frac{b^2}{a^2}\cdot\frac{a^2b^2}{y^3a^2}$$

$$\therefore \quad \rho = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left(1 + \frac{b^4x^2}{a^4y^2}\right)^{\frac{3}{2}}}{\frac{b^4}{a^2}\left(\frac{1}{y^3}\right)}$$

$$\Rightarrow \quad \rho = \frac{\left[a^4y^2 + b^4x^2\right]^{\frac{3}{2}} a^2y^3}{a^6y^3b^4} = \frac{\left(b^4x^2 + a^4y^2\right)^{\frac{3}{2}}}{a^4}$$

$$\Rightarrow \quad \rho = \frac{a^2b^2}{\frac{a^6b^6}{\left(b^4x^2 + a^2y^2\right)^{3/2}}} = \frac{a^2b^2}{p^3} \quad \text{(ii)}$$

Example 8. Find the radius of curvature of the curve $y = e^x$ at the point where it crosses the y axis

Solution: Given curve is $y = e^x$ ....(i)

$$\therefore \quad \frac{dy}{dx} = e^x \; ; \; \frac{d^2y}{dx^2} = e^x$$

$$\therefore \quad \wp = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left\{1 + e^{2x}\right\}^{\frac{3}{2}}}{e^x} \quad \text{.......(ii)}$$

The given curve (ii) crosses y axis when x = 0

$\therefore$ y = 1 i.e it crosses y axis at (0, 1)

$$\therefore \text{ at } (0, 1),\ \rho = \frac{\{1+e^0\}^{\frac{3}{2}}}{e^0} = 2^{\frac{3}{2}} = 2\sqrt{2}$$

Example 9. In the Cycloidx = a(θ + sonθ), y = a(1 – cosθ) prove that

$$\rho = 4a\cos\left(\theta/2\right)$$

Solution: The given Cycloid is

$$x = a(\theta + \sin\theta) \Rightarrow \frac{dx}{d\theta} = a(1 + \cos\theta) = 2a\cos^2\theta/2$$

$$y = a(1 - \cos\theta) \Rightarrow \frac{dy}{d\theta} = a\sin\theta = 2a\sin\theta/2\cos\theta/2$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{2a\sin\theta/2\cos\theta/2}{2a\cos^2\theta/2} = \tan\theta/2$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{1}{2}\sec^2\theta/2\ \frac{d\theta}{dx} = \frac{1}{2}\sec^2\theta/2 \cdot \frac{1}{2a\cos^2\theta/2}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{1}{4}\sec^4\theta/2$$

$$\therefore \quad \rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left(1+\tan^2\theta/2\right)^{\frac{3}{2}}}{\frac{1}{4a}\sec^4\theta/2} = 4a\ \frac{\sec^3\theta/2}{\sec^4\theta/2}$$

$$\therefore \quad \rho = 4a\cos\theta/2$$

Example 10. Find the radius of curvature at the point 't' on the curve x = a cost, y = b sint

Solution: The given curve is x = a cost, y = b sint

$$\therefore \quad x' = \frac{dx}{dt} = -a\sin t,\ y' = \frac{dy}{dt} = b\cos t$$

$$x'' = \frac{d^2x}{dt^2} = -\,a\cos t,\; y'' = \frac{d^2y}{dt^2} = -\,b\sin t$$

$$\therefore\quad \rho = \frac{\left(x'^2 + y'^2\right)^{3/2}}{x'y'' - y'x''} = \frac{\left(a^2\sin^2 t + b^2\cos^2 t\right)^{3/2}}{ab\sin^2 t + ab\cos^2 t}$$

$$\therefore\quad \rho = \frac{\left(a^2\sin^2 t + b^2\cos^2 t\right)^{3/2}}{ab}$$

Example 11. Find the radius of curvature at (p, r) on the curve $p^2 = ar$

Solution: The curve is $p^2 = ar$

Differentiating w.r.t r we get

$$2p\,\frac{dp}{dr} = a \Rightarrow \frac{dp}{dr} = \frac{a}{2p}$$

$$\therefore\quad \frac{dr}{dp} = \frac{2p}{a}$$

$$\therefore\quad \rho = r\,\frac{dr}{dp} = r.\frac{2p}{a} = \frac{p^2}{a}.\frac{2p}{a} = \frac{2p^3}{a^2}$$

Example 12. Find the radius of curvature at the point (p, r) on the following curves

(i) $r^3 = 2ap^2$ (ii) $p^2\,(r^2 + a^2) = r^4$

Solution: (i) Given curve is $r^3 = 2ap^2$

Differentiating w.r.t p we get

$$3r^2\,\frac{dr}{dp} = 4ap \Rightarrow \frac{dr}{dp} = \frac{4ap}{3r^2}$$

$$\therefore\quad \rho = r\,.\frac{dr}{dp} = r.\,\frac{4ap}{3r^2} = \frac{4ap}{3r} = \frac{4a}{3r}\sqrt{\frac{r^3}{2a}}$$

$$\therefore\quad \rho = \frac{2}{3}\sqrt{\frac{2r}{a}}$$

(ii) Given curve is $p^2 (r^2 + a^2) = r^4$ ....(i)

Differentiating w.r.t r we get

$$2rp^2 + r^2\, 2p\frac{dp}{dr} + a^2.2p\frac{dp}{dr} = 4r^3$$

$$\Rightarrow \quad (r^2 + a^2)\, 2p\frac{dp}{dr} = 4r^3 - 2rp^2$$

$$\Rightarrow \quad \frac{dp}{dr} = \frac{4r^3 - 2rp^2}{2p\left(r^2 + a^2\right)} = \frac{2r^3 - rp^2}{p\left(r^2 + a^2\right)}$$

$$\therefore \quad \rho = r\frac{dr}{dp} = r\frac{p(r^2 + a^2)}{2r^3 - rp^2} = \frac{p\left(r^2 + a^2\right)}{2r^2 - p^2}$$

$$\Rightarrow \quad \rho = \sqrt{\frac{r^4}{r^2 + a^2}}\left[\frac{r^2 + a^2}{2r^2 - \frac{r^4}{r^2 + a^2}}\right] \quad \text{Putting the value of p from (i)}$$

$$\Rightarrow \quad \rho = \frac{r^2}{\sqrt{r^2 + a^2}}\left[\frac{r^2 + a^2}{\frac{2r^4 + 2r^2a^2 - r^4}{r^2 + a^2}}\right]$$

$$\Rightarrow \quad \rho = \frac{r^2}{2r^2a^2 + r^4}\left(r^2 + a^2\right)\sqrt{r^2 + a^2}$$

$$\Rightarrow \quad \rho = \frac{\left(r^2 + a^2\right)^{\frac{3}{2}}}{2a^2 + r^2}$$

Example 13. Find the radius of curvature at any point (r, θ) on the cardiode $r = a(1+\cos\theta)$

Solution: Curve is $r = a(1+\cos\theta)$

$$\therefore \quad \frac{dr}{d\theta} = -a\sin\theta\ ; \quad \frac{d^2r}{d\theta^2} = -a\cos\theta$$

$$\therefore \quad \rho = \frac{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{3/2}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\left(\frac{d^2r}{d\theta^2}\right)}$$

$$\Rightarrow \quad \rho = \frac{\left(a^2(1+\cos\theta)^2 + a^2\sin^2\theta\right)^{3/2}}{a^2(1+\cos\theta)^2 + 2a^2\sin^2\theta + a^2(1+\cos\theta)\cos\theta}$$

$$\Rightarrow \quad \rho = \frac{a^3\left\{1+\cos^2\theta + 2\cos\theta + \sin^2\theta\right\}^{3/2}}{a^2\left[1+\cos^2\theta + 2\cos\theta + 2\sin^2\theta + \cos\theta + \cos^2\theta\right]}$$

$$\Rightarrow \quad \rho = \frac{a(2+2\cos\theta)^{3/2}}{3\cos\theta + 2(\cos^2\theta + \sin^2\theta) + 1}$$

$$\Rightarrow \quad \rho = \frac{2a\sqrt{2}\,(1+\cos\theta)^{3/2}}{3(1+\cos\theta)} = \frac{2a\sqrt{2}}{3}(1+\cos\theta)^{1/2}$$

$$\Rightarrow \quad \rho = \frac{2a\sqrt{2}}{3}\sqrt{2\cos^2\theta/2} = \frac{4a}{3}\cos\theta/2$$

Example 14. Find the radius of curvature at any point (r, θ) on the cardiode $r = a(1 - \cos\theta)$

Solution: Similar as example 13.

Example 15. For any curvature prove that $\frac{r}{\rho} = \sin\phi\left(1+\frac{d\phi}{d\theta}\right)$ were ρ is the radius of curvature and $\tan\phi = r\,\frac{d\theta}{dr}$

Solution: We know that

$\psi = \theta + \phi$ ....(i)

Differentiating both sides of (i) w.r.t θ, we get

$$\frac{d\psi}{d\theta} = 1 + \frac{d\phi}{d\theta}$$

$$\Rightarrow \quad \sin\phi\frac{d\psi}{d\theta} = \sin\phi\left(1+\frac{d\phi}{d\theta}\right) \quad \text{Multiplying both sides by } \sin\phi$$

$$\Rightarrow \quad r\frac{d\theta}{ds}.\frac{d\psi}{d\theta} = \sin\phi\left(1+\frac{d\phi}{d\theta}\right) \qquad \left[\because \sin\phi = r\frac{d\theta}{ds}\right]$$

$$\Rightarrow \quad r\frac{d\psi}{ds} = \sin\phi\left(1+\frac{d\phi}{d\theta}\right)$$

$$\Rightarrow \quad \frac{r}{\rho} = \sin\phi\left(1+\frac{d\phi}{d\theta}\right) \qquad \because \rho = \frac{ds}{d\psi}$$

12.3 Curvature at the Origin

(i) Substitution Method

We know that the value of radius of curvature at any point (x, y) is

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} \quad \text{..............(i) provided } \frac{d^2y}{dx^2} \neq 0$$

$\therefore$ Substituting x = 0, y = 0 in the value of ρ obtained the radius of curvature at the origin can be found.

(ii) Expansion Method

When the equation of the curve can be expanded in power of x, then we can write

$$y = px + \frac{qx^2}{2!} + ...$$

Then we find that the curve passes through the origin and

$$p = \left(\frac{dy}{dx}\right)_{x=0,y=0} \quad \text{and} \quad q = \left(\frac{d^2y}{dx^2}\right)_{x=0,y=0} \quad \text{and comparing with}$$

$$y = f(0) + x\, f^1(0) + \frac{x^2}{2!} f^2(0) \; ....$$

we get form (i) above

$$p \text{ (at the origin)} = \frac{\left(1+p^2\right)^{3/2}}{q}$$

(iii) Newtonian Method

If the curve passes through origin and the tangent at the origin is x-axis, then we have x=0, y=0 and $\frac{dy}{dx}$ = 0 at the origin.

∴ The expansion of y by Maclaurn's Theorem reduces to

$$y = 0 + 0\,x + \frac{q}{2!}x^2 + \ldots\ldots..$$

Dividing each term by x^2 we get

$$q = \frac{2y}{x^2}$$

Taking limit as $x \to 0$ we have

$$q = \lim_{x \to 0}\left(\frac{2y}{x^2}\right) \quad \ldots\ldots(ii)$$

Since ρ (at the origin) $= \left(\dfrac{1+p^2}{q}\right)^{3/2}$

and p=0 at the origin, we get from (ii) above

$$p = \frac{1}{q} = \lim_{x \to 0}\left(\frac{x^2}{2y}\right)$$

Similarly if the curve passes through the origin and the tangent at the origin is y-axis then by interchanging x and y in the last result we get

$\rho = \lim_{y \to 0}\left(\dfrac{y^2}{2x}\right)$ [it should be noted here that as $x \to 0$, $y \to 0$ also]

12.4 Curvature at the Pole

We know $x = r\cos\theta$ and $y = r\sin\theta$

At the pole (if initial line is tangent to the curve at the pole

$$\rho = \lim_{x \to 0}\left(\frac{x^2}{2y}\right)$$

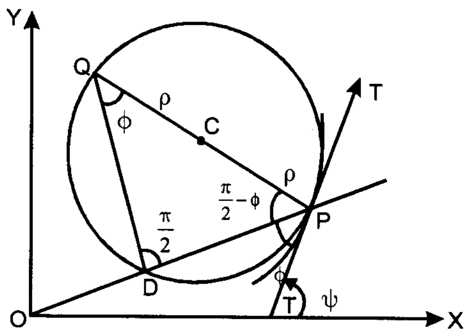
$$= \lim_{\theta \to 0}\left[\frac{r^2\cos^2\theta}{2r\sin\theta}\right]$$

$$= \lim_{\theta \to 0}\left[\frac{r}{2\theta}.\frac{\theta}{\sin\theta}.\cos^2\theta\right]$$

$$\text{or } \rho = \lim_{\theta \to 0}\left[\frac{r}{2\theta}\right]$$

$$\left[\because \theta \to 0 \; \frac{\theta}{\sin\theta} \to 1 \text{ and } \cos^2\theta \to 1\right]$$

12.5 Chord of Curvature through the origin (pole)



Let PD be any chord of a circle of curvature at P and which passes through the origin O.

PT is the tangent at P and PQ is the normal at P of length 2ρ.

DQ is joined and since angle in a semi circle is $\pi/2$, $\angle QDP = \pi/2$

Let $\angle QPT = \phi$ and since angle between the lines is equal to the angle between their penpendicular on the lines, hence $\angle PQD = \phi$

Now from ΔPQD, we have $\frac{PD}{PQ} = \sin\phi$

$$\Rightarrow PD = PQ \sin\phi$$

i.e chord $PD = 2\rho \sin\phi$ ....(i)

(i) Cartesian Form

If the equation of the curve be $y = f(x)$ then we have

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left(1+y_1^{\,2}\right)^{3/2}}{y_2}$$

Where $\tan\psi = \frac{dy}{dx} = y_1,\ y_2 \neq 0$

$$\therefore \sin\psi = \frac{1}{\operatorname{cosec}\psi} = \frac{1}{\sqrt{1+\cot^2\psi}} = \frac{\tan\psi}{\sqrt{1+\tan^2\psi}}$$

$$\text{i.e. } \sin\psi = \frac{y_1}{\sqrt{1+y_1^2}}$$

$$\therefore \cos\psi = \frac{1}{\sqrt{1+y_1^2}}$$

Also $x = r\cos\theta$ and $y = r\sin\theta$ and $r = \sqrt{x^2+y^2}$

Now we know that $\psi = \theta+\phi \Rightarrow \phi = \psi - \theta$

$\therefore \sin\psi = \sin(\psi-\theta) = \sin\psi \cos\theta - \cos\psi \sin\theta$

$$\text{i.e } \sin\psi = \frac{y_1}{\sqrt{1+y_1^2}}\cdot\frac{x}{r} - \frac{1}{\sqrt{1+y_1^2}}\cdot\frac{y}{r}$$

$$= \frac{xy_1 - y}{r\sqrt{1+y_1^2}}$$

Hence chord PD $= 2\rho\sin\phi$

$$= \frac{2(1+y_1^2)^{3/2}}{y_2}\cdot\frac{xy_1 - y}{r\sqrt{1+y_1^2}}$$

$$= \frac{2(xy_1 - y)(1+y_1^2)}{y_2\sqrt{x^2+y^2}} \quad \text{...........(A)}$$

(ii) Polar Form

If the equation of the curve be $r = f(\theta)$

We have $\sin\phi = r\frac{d\theta}{ds}$ where $ds = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2} = \sqrt{r^2+r_1^2}$

$$\text{Then } \rho = \frac{(r^2+r_1^2)^{3/2}}{r^2+2r_1^2 - rr_2}$$

Hence chord PD = $2\rho\ \sin\phi = 2\ \frac{\left(r^2+r_1^2\right)^{3/2}}{r^2+2r_1^2-r\,r_2}\cdot\frac{r}{\sqrt{r^2+r_1^2}}$

$$= \frac{2r\left(r^2+r_1^2\right)}{r^2+2r_1^2-r\,r_2} \quad ....(B)$$

(iii) Pedal Form

If the equation of the curve be $p = f(r)$

then $\frac{dp}{dr} = f'(r)$ and $\rho = r\ \frac{dr}{dp} = \frac{r}{f'(r)}$

Hence chord PD = $2\rho\sin\phi = 2.\frac{r}{f'(r)}.\frac{p}{r}$ $\quad\because p = r\sin\phi$

$$= \frac{2p}{f'(r)} \quad .....(C)$$

Remarks

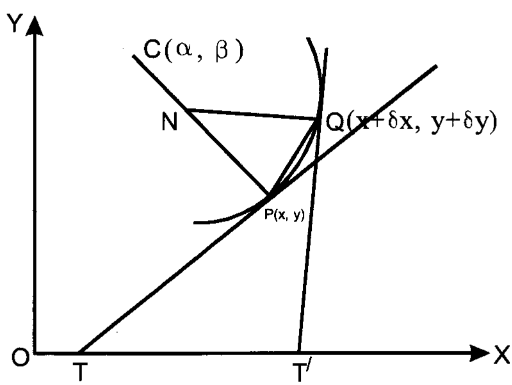
(a) If the chord is parallel to x-axis, then angle ϕ between chord and the tangent is equal to ψ

$\therefore$ Chord $= 2\rho\sin\phi = 2\rho\sin\psi$

(b) If the chord is parallel to y-axis, then the angle ϕ between chord and the tangent is equal to $\frac{\pi}{2}-\psi$

$\therefore$ Chord $= 2\rho\sin\phi = 2\rho\sin\left(\frac{\pi}{2}-\psi\right) = 2\rho\cos\psi$

12.6 Centre of Curvature



Let P(x, y) and Q (x+δx, y+δy) be two neighbouring points on the curve. Let C be the limiting position of N, the point of intersection of the normals at P and Q to the curve when Q$\to$P

Let C (α, β) be the centre of curvature.

The equation of the normal at P is

$$(Y - y)\ \phi(x) + (X - x) = 0 \ \text{...........(i) where } \phi(x) = \frac{dy}{dx}$$

The equation of the normal at Q is

$$\{Y - (Y + \delta y)\}\ \phi(x+\delta x) + \{X - (x + \delta x)\} = 0 \ \text{...........(ii)}$$

Subtracting (i) from (ii) we get

$$(Y - y)\ \{\phi(x) + \delta x) - \phi(x)\} - \phi(x + \delta x)\ \delta y - \delta x = 0 \ \text{......(iii)}$$

Now as Q$\to$P along the curve, $\delta x\to 0$, N$\to$C i.e $y\to\beta$

$$\therefore \frac{(x+\delta x)-\phi(x)}{\delta x}\to\frac{d}{dx}\{\phi(x)\}=\frac{d^2y}{dx^2}$$

Form (ii) we have

$$(\beta-y)\frac{d^2y}{dx^2}-\phi(x)\frac{dy}{dx}-1=0$$

or $$(\beta-y)\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2-1=0 \qquad \because \phi(x) = \frac{dy}{dx}$$

or $$\beta=y+\frac{1+\left(\frac{dy}{dx}\right)^2}{d^2y\big/dx^2}$$

As C (α, β) lies on the normal at P

$\therefore$ from (i) we get

$$(\beta-y)\frac{dy}{dx}+(\alpha-x)=0$$

or $$\alpha=x-(\beta-y)\frac{dy}{dx} = x-\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}{\frac{d^2y}{dx^2}}\cdot\frac{dy}{dx}$$

Note: From above we can write down the equation of the circle of curvature as

$$(x - \alpha)^2 + (y - \beta)^2 = \rho^2$$

The locus of the centre of curvature is called its evolute.

Illustrative Examples

Example 1. Find the radius of curvature at the origin of the curve $y = x^4 - 4x^3 - 18x^2$

Solution: The given curve is $y = x^4 - 4x^3 - 18x^2$ ......(i)

Differentiating both sides w.r.t x we get

$$\frac{dy}{dx} = 4x^3 - 12x^2 - 36x$$

and $$\frac{d^2y}{dx^2} = 12x^2 - 24x - 36$$

At the origin (0, 0) $\frac{dy}{dx} = 0$, $\frac{d^2y}{dx^2} = -36$

$$\therefore \rho \text{ (at the origin)} = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(1+0)^{3/2}}{-36}$$

$$= \frac{1}{36} \text{ (numercially)}$$

Example 2. Obtain the radii of curvature of the curve $a(y^2 - x^2) = x^3$ at the origin

Solution: The given curve is $a(y^2 - x^2) = x^3$

or $$y^2 = \frac{x^3}{a} + x^2$$

or $$y = \pm x\sqrt{1+\frac{x}{a}} = \pm x\left(1+\frac{x}{a}\right)^{1/2}$$

or $$y = \pm x\left(1+\frac{1}{2}.\frac{x}{a}+.....\right)$$ [by Binomial theorem] or $\quad y = \pm\left(x + \frac{1}{2}.\frac{x^2}{a} + ...\right)$ which is of the form $y = px + \frac{qx^2}{2!} + ...$

$\therefore \quad p = 1,\ q = \frac{1}{a}$ or $p = -1$ and $q = -\frac{1}{a}$

Now ρ (at the origin) $= \dfrac{(1+p^2)^{3/2}}{q}$

$$= \frac{(1+1)^{3/2}}{1/a} = 2a\sqrt{2}$$

and ρ (at the origin) $= \dfrac{(1+1)^{3/2}}{-1/3} = -2a\sqrt{2}$

Required values of ρ are $\pm\ 2a\sqrt{2}$

Example 3. Find the radius of curvature at the origin for the curve $x^3 + y^3 - 2x^2 + 6y = 0$

Solution: The curve passes through (0, 0) and the tangent there at is $y = 0$ (which is obtained by equating the lowest degree term in the equation to zero)

ρ (at the origin) $= \lim\limits_{x\to 0}\left(\dfrac{x^2}{2y}\right)$

Now the given curve is $x^3 + y^3 - 2x^2 + 6y = 0$

Dividing each term by 2y we get

$$x\left(\frac{x^2}{2y}\right) + \frac{1}{2}y^2 - 2\left(\frac{x^2}{2y}\right) + 3 = 0$$

taking limits as $x \to 0$ and $y \to 0$ we get

$$0 + 0 - 2\rho + 3 = 0 \text{ or } \rho = 3/2$$

Example 4. Use Newtonian method to find the radius of curvature at the origin for the Cycloid

$$x = a\,(\theta + \sin\theta),\ y = a\,(1 - \cos\theta)$$

Solution: Curve is $x = a\,(\theta + \sin\theta),\ y = a\,(1 - \cos\theta)$

$$\therefore \quad \frac{dx}{d\theta}=a(1+\cos\theta)\,;\ \frac{dy}{d\theta}=a\sin\theta$$

$$\therefore \quad \frac{dy}{dx}=\frac{dy/d\theta}{dx/d\theta}=\frac{a\sin\theta}{a(1+\cos\theta)}=\frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}=\tan\theta/2$$

At the origin when $x = 0$, $y = 0$ hence $\theta = 0$

$\therefore \frac{dy}{dx} = 0$ at the origin

Hence x-axis is the tangent at the origin

$$\therefore \rho \text{ (at the origin)} = \lim_{x\to 0}\left(\frac{x^2}{2y}\right)$$

$$= \lim_{\theta\to 0}\frac{a^2(\theta+\sin\theta)^2}{2a(1-\cos\theta)}$$

$\therefore$ as $x\to 0$, $y\to 0$, $\theta\to 0$ and this limit is of the form $\frac{0}{0}$

$$\text{Hence } \rho \text{ (at the origin)} = \lim_{\theta\to 0}\left[\frac{a}{2}\frac{2(\theta+\sin\theta)(1+\cos\theta)}{\sin\theta}\right] \quad \ldots \frac{0}{0}$$

$$= \lim_{\theta\to 0}\left[\frac{a(\theta+\sin\theta)(-\sin\theta)+(1+\cos\theta)^2}{\cos\theta}\right]$$

$$= 4a$$

Example 5. Show that the chord of curvature through the pole of the curve $r = ae^{m\theta}$ is $2r$

Solution: The given curve is $r = ae^{m\theta}$ ......(i)

Differentiating both sides w.r.t θ we get

$$\frac{dr}{d\theta}=mae^{m\theta}=mr \text{ from (i)}$$

$$\therefore \quad \tan\phi = r=\frac{d\theta}{dr}=\frac{r}{mr}=\frac{1}{m}$$

$$\therefore \quad \sin\phi = \frac{1}{\operatorname{cosec}\phi}=\frac{1}{\sqrt{1+\cot^2\theta}}=\frac{\tan\phi}{\sqrt{1+\tan^2\phi}}=\frac{1}{\sqrt{1+m^2}}$$

From $p = r\sin\phi$ we have the pedal equation of the curve as

$$p = \frac{r}{\sqrt{1+m^2}}$$

$$\therefore \quad \frac{dp}{dr} = \frac{1}{\sqrt{1+m^2}}$$

$$\therefore \quad \rho = r\frac{dr}{dp} = r\sqrt{1+m^2}$$

$\therefore$ Chord of curvature $= 2\rho\sin\phi$

$$= 2r\sqrt{1+m^2}\,.\,\frac{1}{\sqrt{1+m^2}}$$

$$= 2r$$

Example 6. Find the chord of curvature through the pole of the cardiode $r = a(1 + \cos\theta)$

Solution: The given curve is $r = a(1 + \cos\theta)$ ......(i)

Differentiating w.r.t θ we get

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\therefore \tan\phi = r\frac{d\theta}{dr} = -\frac{r}{a\sin\theta} = -\frac{a(1+\cos\theta)}{a\sin\theta}$$

$$\text{or } \tan\phi = -\frac{2\cos^2\theta/2}{2\sin\theta/2\cos\theta/2} = -\cot\theta/2$$

$$= \tan\left(\frac{\pi}{2}+\theta/2\right)$$

$$\therefore \theta = \frac{\pi}{2}+\frac{\theta}{2} \quad ......(ii)$$

Also from $p = r\sin\phi$, then pedal equation of the given curve is $p = r\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right) = r\cos\theta/2$

$$\text{or } 2p^2 = r^2\, 2\cos^2\theta/2 = r^2(1+\cos\theta) = r^2.\frac{r}{a} \text{ from (i)}$$

$$= \frac{r^3}{a}$$

$$\text{or } 2ap^2 = r^3$$

Differentiating w.r.t r we get

$$4ap\frac{dp}{dr} = 3r^2 \text{ ....(iii)}$$

$$\therefore\ \rho = r\frac{dr}{dp} = r\frac{4ap}{3r^2} \text{ from (iii)}$$

$$= \frac{4ap}{3r}$$

$$\therefore \text{ Chord of curvature } = 2\rho\sin\phi$$

$$= 2.\frac{4ap}{3r}.\sin\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$$= \frac{8ap}{3r}\cos\frac{\theta}{2}$$

$$= \frac{8a}{3r}\cos\frac{\theta}{2}\, r\cos\frac{\theta}{2} \qquad \left[\because p = r\cos\frac{\theta}{2}\right]$$

$$= \frac{8a}{3}\cos^2\frac{\theta}{2} \qquad \left[\because 2\cos^2\frac{\theta}{2} = 1+\cos\theta\right]$$

$$= \frac{4}{3}a\,(1+\cos\theta) = \frac{4}{3}r$$

Example 7. In the curve $y = a\log\sec\left(\frac{x}{a}\right)$, prove that the chord of curvature parallel to y-axis is of constant length.

Solution: The curve is $y = a\log\sec\left(\frac{x}{a}\right)$ ....(i)

Differentiating w.r.t x we get

$$\frac{dy}{dx} = a\frac{1}{\sec\left(x/a\right)}\sec\left(\frac{x}{a}\right)\tan\left(\frac{x}{a}\right).\frac{1}{a}$$

$$= \tan\left(\frac{x}{a}\right)$$

$$\therefore\ \frac{d^2y}{dx^2} = \frac{1}{a}\sec^2\left(\frac{x}{a}\right)$$

$$\therefore \quad \tan\psi = \frac{dy}{dx} = \tan\left(\frac{x}{a}\right)$$

or $\psi = \dfrac{x}{a}$

Also $$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left[1+\tan^2\left(\frac{x}{a}\right)\right]^{\frac{3}{2}}}{\frac{1}{a}\sec^2\left(\frac{x}{a}\right)}$$

$$= \frac{a\sec^3\left(\frac{x}{a}\right)}{\sec^2\left(\frac{x}{a}\right)} = a\sec\left(\frac{x}{a}\right)$$

Chord of curvature parallel to y axis = $2\rho\cos\psi$

$$= 2a\sec\left(\frac{x}{a}\right)\cos\left(\frac{x}{a}\right) \qquad \left[\because \psi = \frac{x}{a}\right]$$

$$= 2a, \text{ a constant}$$

Example 8. Find the coordinates of the centre of curvature for the point (x, y) on the parabola $y^2 = 4ax$

Soution: Here $\dfrac{dy}{dx} = \dfrac{2a}{y} = \dfrac{2a}{\sqrt{4ax}} = \sqrt{\dfrac{a}{x}}$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{1}{2}a^{1/2}x^{-3/2}$$

$$\therefore \quad \alpha = x - \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}}\frac{dy}{dx}$$

$$= x - \frac{\left(1+\frac{a}{x}\right)\sqrt{\frac{a}{x}}}{-\frac{1}{2}a^{1/2}x^{-3/2}}$$

or $\alpha = x + 2(x + a) = 3x + 2a$

$$\beta = y + \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]}{\frac{d^2y}{dx^2}} = y + \frac{\left(1+\frac{a}{x}\right)}{-\frac{1}{2}a^{1/2}x^{-3/2}}$$

or $\beta = y - 2a^{-1/2}x^{1/2}(x+a) = 2a^{1/2}x^{1/2} - 2a^{-1/2}x^{1/2}(x+a)$

$\because y^2 = 4ax$

Hence the required centre of curvature is given by

$\alpha = 3x + 2a$; $\beta = 2a^{1/2}x^{1/2} - 2a^{-1/2}x^{1/2}(x+a)$

Exercises

1. Find the radius of curvature at any point (x, y) for the following curves –

   (i) $e^{y/a} = \sec\left(\frac{x}{a}\right)$ (ii) $xy = c^2$ (iii) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

   (iv) $y = x^3 - 2x^2 + 7x$ at the origin

   (v) $y = xe^{-x}$ at the maximum point

2. Find the radius of curvature at the indicated point for each of the following curves –

   (i) $x = a\cos\phi$, $y = b\sin\phi$ at ϕ

   (ii) $x = a\sec\phi$, $y = b\tan\phi$ at ϕ

   (iii) $x = a(\cos t + t\sin t)$ $y = a(\sin t - t\cos t)$ at t

   (iv) $x = a(\theta + \sin\theta)$; $y = a(1 - \cos\theta)$ at $\theta=0$

3. Find the radius of curvature at any point (s, ψ) on the following curves:

   (i) $s = 4a\sin\psi$ (ii) $s = 8a\sin^2\frac{\psi}{6}$ (iii) $s = c\tan\psi$

   (iv) $s = c\log\sec\psi$

4. Find the radius of curvature at any point (r, θ) for the following curves

   (i) $r = a\sec^2\theta/2$ (ii) $r^3 = a^3\cos 3\theta$ (iii) $r^2\cos 2\theta = a^2$

   (iv) $r = a(\theta + \sin\theta)$ at $\theta = 0$

5. Find the radius of curvature at the origin of the following curves

   (i) $2x^2 - xy + y^2 - y = 0$ (ii) $3x^2 + xy + y^2 - 4x = 0$

(iii) $x^2 + 6y^2 + 2x - y = 0$ (iv) $y = x^4 - 4x^3 - 18x^2$

(v) $x^4 + y^2 = 6a(x + y)$ (vi) $y^2 = x^2(a + x)/(a-x)$

(vii) $y^2 - 2xy - 3x^2 - 4x^3 - x^2y^2 = 0$

6. Find the radius of curvature at any point on the curves

(i) $p = a(1 + \sin\psi)$ (ii) $p = a \operatorname{cosec}\psi$ (iii) $p^2 + a^2 \cos 2\psi = 0$

7. Find the chord of curvature through the pole of the curves:

(i) $r = a^2 \cos 2\theta$ (ii) $r^2 \cos 2\theta = a^2$ (iii) $r = ae^{\theta\cot\alpha}$

8. Prove that the radius of curvature at the vertex of the parabola is equal to its latus rectum.

9. Show that the chord of curvature through the pole for the curve $p = f(r)$ is $\dfrac{2f(r)}{f'(r)}$

10. In the cycloid $x = a(\theta+\sin\theta)$; $y = a(1-\cos\theta)$ prove that $\rho = 4a\cos\theta/2$

11. In the curve $r^n = a^n \cos n\theta$, show that the radius of curvature varies inversely as the (n–1)th power of the radius vector

12. Find the curvature of $y = x \log x$ at the minimum value of y

13. Show that in the curve $y - 3xy - 4x^2 + x^3 + x^4y + y^5 = 0$, the radii of curvature at the origin are $\frac{85}{2}\sqrt{17}$ and $5\sqrt{2}$

14. Find the center of curvature of the following curves at the point (x, y)

(i) $x^2 = 4ay$ (ii) $x^{2/3} + y^{2/3} = a^{2/3}$ (iii) $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

(iv) $x = at^2$, $y = 2at$ (v) $x = a(\theta-\sin\theta)$; $y = a(1-\cos\theta)$

15. Find the centre of curvature of the following curves at the points indicated

(i) $xy = x^2 + 4$ at (2, 4) (ii) $y = \sin^2 x$ at (0, 0)

(iii) $y = x^3 + 2x^2 + x + 1$ at (0, 1)

(iv) $x = e^{-2t}\cos 2t$, $y = e^{-2t}\sin 2t$ at $t = 0$

16. Show that in any curve,

(i) $\rho = \left[\left(\dfrac{dx}{d\psi}\right)^2 + \left(\dfrac{dy}{d\psi}\right)^2\right]^{1/2}$ (ii) $\dfrac{1}{\rho} = -\dfrac{d^2x/ds^2}{dy/ds} = \dfrac{d^2y/dx^2}{dx/ds}$

17. Show that in the cycloid $s^2 = 8ay$

$$\rho = 4a\sqrt{1 - y/2a}$$

18. Find the radius of curvature of the curve $x^2 + 4xy - 2y^2 = 10$ at the point (2, 1)

19. Find the radius of curvature of the curve

$x = a(\cos\theta + \theta\sin\theta)$, $y = a(\sin\theta - \theta\cos\theta)$ at $x = \theta = \pi/4$

13

Asymtotes

Introduction

We know that there are certain curves which are limited in extent, for example circle, ellipse etc where as there are curves like parabola or hyperbola which extend to infinity. In the latter case, the tanget drawn at any point of the curve, if its point of contact moves further and further from the origin, then three cases may arise i.e (i) the tangent may go on moving away from the origin or (ii) the distance of the tangent from the origin may keep oscillating i.e sometimes increasing and sometimes decreasing or (iii) the tangent may tend to a definite straight line, at a definite distance from the origin, which we called the asymtote.

13.1 Definition

A straight line at a finite distance from the origin to which the tangent to a curve tends, as the point of contact tends to infinity is called an asymtote to the curve. In otherwords, a straight line which touch the curve at infinity but is situated at a finite distance from the origin is called an aymtote to the curve.

13.2 Working Rule to Obtain the Equations of the Oblique Asymtotes of a given curve

If the straight line $y = mx + c$ ....(i)

is to be an asymptote to the curve $f(x, y) = 0$ ....(ii)

then $f(x, y) = 0$ being a rational, algebraic function of nth degree in x and y, the coefficients of two of the highest powers of x in the equation obtained by substituting (i) in (ii) must be seperately zero.

13.3 Shorter Method to Obtain Equations of the Oblique Asymtotes to a given cruve

If the given equation of the curve is of nth degree, then by putting x=1 and y=m in the highest degree terms, thus getting $\phi_n(m)$.

Equate $\phi_n(m)$ to zero and solve for m

Now putting x = 1 and y = m in the (n−1)th degree terms in the given equation, thus getting $\phi_{n-1}(m)$

Also we know $c = -\dfrac{\phi_{n-1}(m)}{\phi_n'(m)}$

From this we can obtain the value of c such as $c_1, c_2, c_3 \ldots c_n$ corresponding to the values $m_1, m_2, m_3, \ldots m_n$. Then the asymptotes are

$y = m_1x + c_1$, $y = m_2x + c_2$ etc

13.4 Non-existence of Asymptotes

If one or more values of m obtained from $\phi_n(m)$ are such that they make $\phi_n^1(m) = 0$ whereas $\phi_{n-1}(m) = 0$ then from

$$c = -\frac{\phi_{n-1}(m)}{\phi_n^1(m)} \text{ we get } c = +\infty \text{ or } -\infty$$

and this corresponds to the case where the tangent goes farther and farther from the origin as $n \to \infty$

13.5 Two parallel Asymptotes

If any value of m say m_1 found from $\phi_n(m) = 0$ makes $\phi_n'(m)=0$ and $\phi_{n-1}(m-1) = 0$ [which happens only if $\phi_n(m) = 0$ has repeated roots], then the equation $C\phi_n(m) + \phi_{n-1}(m) = 0$ reduces to the identity c.0+0 = 0

To determine c in such cases we have the equation

$$\left(\frac{c^2}{2}\right)\phi_n''(m)+C\phi'_{n-1}(m)+\phi_{n-2}(m)=0 \quad \ldots\ldots(i)$$

Hence corresponding to this value m_1 of m we shall have two values of C from (i) i.e we shall get a pair of parallel asymptotes

Hence the rule for finding two parallel asymptotes is:

Substituting y = mx + c in the given equation of the curve and equate to zero the coefficient of highest powers of x i.e $\phi_n(m) = 0$. If any value of m is obtained from $\phi_n(m)=0$ makes the coefficient of x^{n-1} identically zero, then the corresponding values of c are determined from the equation obtained by putting the coefficient of x^{n-1} equal to zero.

The method can be generalised in case of more than three parallel asymptotes.

13.6 Three or more Parallel Asymptotes

In the above if we find the $\phi_n(m)=0$ has three equal roots, then there are three parallel asymptotes. In such case C cannot be obtained from equation (i) but is obtained from the cubic equation

$$\frac{C^3}{3!}\phi_n'''(m)+\frac{C^2}{2!}\phi_{n-1}''(m)+C\phi_{n-2}'(m)+\phi_{n-3}(m)=0$$

Hence the rule for finding three parallel asymtotes is:

Substituting $y = mx + c$ in the given equationof the curve and equate to zero, the coefficient of highest powers of x i.e. $\phi_n(m)=0$. If three values obtained from $\phi_n(m)=0$ be equal, then there are three parallel asymptotes. Also if such value of $\phi_n(m)=0$ will make the coefficient of x^{n-1} and x^{n-2} zero, then the corresponding values of the determined from the equation obtained by the coefficient of x^{n-1} equal to zero.

The method can be generalised in case more than three parallel asymptotes.

13.7 Asymptotes Parallel to the axes

(a) Asymptotes Parallel to x-axis

Asymptote or asymptotes parallel to x-axis can be obtained by equating the coefficient of the highest power of x to zero, provided that the coefficient of the highest power of x is not constant.

(b) Asymptotes Parallel to y-axis

Asmptote or asymptotes parallel to y-axis can be obtained by equating the coefficient of the highest power of y to zero, provided that the coefficient ofthe highest power of y is not constant.

13.8 Asymptotes to Polar Curves

For finding the asymptotes to polar curves, the method is as follows

(i) Put the equation of the curve in the form $\frac{1}{r}=f(\theta)$

(ii) Find the roots of the equation $f(\theta) = 0$

(iii) The asymptote corresponding to a root α of the equation $f(\theta) = 0$ is given by

$$r \sin(\theta - \alpha) = \frac{1}{f'(\alpha)}$$

Illustrative Examples

Example 1. Find the asymptotes of the curve $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$ (NEHU 2016)

Solution: Let $y = mx + c$ ....(i) be an asymptotes of the given curve

Substituting $y = mx + c$ in the equation of the given curve we get

$$x^3 + 3x^2(mx + c) - 4(mx + c)3 - x + (mx + c) + 3 = 0$$

or $x^3 + 3mx^2 + 3cx^2 - 4m^3x^3 - 4c^3$

$$- 12m^2x^2c - 12mxc^2 - x + mx + c + 3 = 0$$

or $x^3(1 + 3m - 4m^3) + x^2(3c - 12m^2c)$

$$+ x(-12mc^2 - 1 + m) + (c + 3 - 4c^3) = 0 \text{ ......(ii)}$$

Equating the coefficients of highest power of x in (ii) to zero we get

$$1 + 3m - 4m^3 = 0$$

or $1 + 4m - m - 4m^3 = 0$

or $(1 - m) + 4m(1 - m^2) = 0$

or $(1 - m) + 4m(1 - m)(1 + m) = 0$

or $(1 - m)(1 + 4m(1 - m)) = 0$

or $(1 - m)(1 + 2m)^2 = 0$

or $m = 1, -\frac{1}{2}, -\frac{1}{2}$

Equating the coefficient of highest power of x to zero in (ii) we get

$$3C - 12m^2C = 0$$

or $C(3 - 12m^2) = 0$ ....(iii)

When $m = -\frac{1}{2}$ we have $C(0) = 0$. Where from C cannot be determined.

Again equating the coefficient of x in (ii) to zero we get

$$-12\,mC^2 - 1 + m = 0$$

or $\quad 6\,C^2 - \frac{3}{2} = 0$ when $m = -\frac{1}{2}$

i.e $\quad C^2 = \frac{1}{4}$ or $C = \pm\frac{1}{2}$

When $m = 1$ we have from (iii) $C(-9) = 0 \Rightarrow C = 0$

$\therefore$ The corresponding asymptotes are

$$y = x \text{ and } y = -\frac{1}{2}x \pm \frac{1}{2}$$

i.e $\quad y = x$ and $x + 2y = \pm 1$

Example 2. Find the asymptotes of the following curve $x^3 + 2x^2y - xy^2 - 2y^3 + xy - y^2 - 1 = 0$

Solution: Given Curve is

$$x^3 + 2x^2y - xy^2 - 2y^3 + xy - y^2 - 1 = 0 \quad .....(i)$$

Let $y = mx + c$ be an asymptote to (i). Then putting $y = mx+c$ in (i) we get

$$x^3 + 2x^2(mx + c) - x(mx + c)^2 - 2(mx + c)^3 + x(mx + 1)$$
$$- (mx + c)^2 - 1 = 0$$

or $\quad x^3 + 2mx^3 + 2cx^2 - m^2x^3 - c^2x - 2mcx^2 - 2m^3x^3$
$$- 2c^3 - 6m^2x^2c - 6mxc^2 + mx^2 + cx$$
$$- m^2x^2 - c^2 - 2mcx - 1 = 0$$

or $\quad x^3(1 + 2m - m^2 - 2m^3) + x^2(2c - 2mc - 6m^2c + m - m^2)$
$$+ x(-c^2 - 6mc^2 - 3mc)$$
$$+ (c^2 - 2c^3 - 1) = 0 \quad ......(ii)$$

Equating the coefficients of x^3 and x^2 to zero we get

$$1 + 2m - m^2 - 2m^3 = 0 \quad ....(iii)$$

and $\quad 2c - 2mc - 6m^2c + m - m^2 = 0 \quad .....(iv)$

From (iii) $(1 - m^2) + 2m(1 - m^2) = 0$

$\Rightarrow \quad (1 - m^2)(1 + 2m) = 0 \Rightarrow m = 1, -1, -\frac{1}{2}$

from (iv) we have

$$2c\,(1 - m - 3m^2) = m^2 - m$$

or $\quad c = \dfrac{m^2 - m}{2(1 - m - 3m^2)}$

When $m = 1$, $c = \dfrac{1-1}{2(1-1-3)} = 0$

When $m = -1$, $c = \dfrac{1+1}{2(1+1-3)} = -1$

When $m = -\dfrac{1}{2}$, $c = \dfrac{\frac{1}{4}+\frac{1}{2}}{2\left(1+\frac{1}{2}-\frac{3}{4}\right)} = \dfrac{3/4}{3/2} = \dfrac{1}{2}$

Substituting these values of m and c in y = mcx + c one by one we get the required asymptotes respectively as

$$y = x \;;\; y = -x - 1 \;;\; y = -\frac{1}{2}x + \frac{1}{2}$$

i.e $\quad y = x \;;\; x + y + 1 = 0$ i.e $x + 2y - 1 = 0$

Example 3. Find the asymptotes of $x^3 + y^3 - 3axy = 0$

Solution: Let $y = mx + c$ ....(i) be an asymptotes to the given curve $x^3 + y^3 - 3axy = 0$ ....(iii)

Putting x = 1 and y = m in the highest degree terms of (ii) we get $\phi_n(m) = 1 + m^3$ or $\phi_n'(m) = 3m^2$ ....(iii)

Equating $\phi_n(m) = 0$ we get $m^3 + 1 = 0$

i.e $m = -1$

Again putting x=1 and y=m in the second degree terms in (ii) we get

$\phi_{n-1}(m) = -3am$ ....(iv)

Also $c = -\dfrac{\phi_{n-1}(m)}{\phi_n'(m)} = \dfrac{3am}{3am^2} = \dfrac{a}{m}$

When $m = -1$, $c = -a$

Hence the required asymptote from (i) is

$y = -x - a$

or $\quad x + y + a = 0$

Example 4. Find the asymptotes of $2x^3 + 3x^2y - 3xy^2 - 2y^3 + 3x^2 - 3y^2 + y - 3 = 0$

Solution: The given curve can be written as

$(2x^3 + 3x^2y - 3xy^2 - 2y^3) + (3x^2 - 3y^2) + (y - 3) = 0$ ...(ii)

Let $y = mx + c$ .....(ii) be an asymptote to the given curve

Putting x=1 and y=m in the highest degree terms of (i)

We get $\phi_n(m) = 2 + 3m - 3m^2 - 2m^3$

$\phi_n^1(m) = 3 - 6m - 6m^2$ ....(iii)

Equating $\phi_n(m)$ to zero, we get

$2 + 3m - 3m^2 - 2m^3 = 0$

or $2 + 2m + m - m^2 - 2m^2 - 2m^3 = 0$

or $2(1 - m^2) + 2m(1 - m^2) + m(1-m) = 0$

or $(1 - m)\,[2(1 + m) + 2m(1+m) + m] = 0$

or $(1 - m)(2m^2 + 5m + 2) = 0$

or $(1 - m)(2m + 1)(m + 2) = 0$

or $m = 1, \ -\frac{1}{2}, -2$

Again putting x=1 and y=m in the next highest degree terms i.e the second degree terms of (i) we have

$\phi_{n-1}(m) = 3 - 3m^2$ ....(iv)

$$\text{Also } C = -\frac{\phi_{n-1}(m)}{\phi'_n(m)} = \frac{3-3m^2}{3-6m-6m^2} \text{ by (iii) and (iv)}$$

$$= \frac{3(m^2-1)}{3(1-2m-2m^2)}$$

Where $m = 1, c = 0$

$$\text{When } m = -\frac{1}{2},\ c = \frac{3\left(\frac{1}{4}-1\right)}{3\left(1+1-\frac{1}{2}\right)} = \frac{-3/4}{3/2} = -\frac{1}{2}$$

$$\text{When } m = -2,\ c = \frac{3(4-1)}{3(1+4-8)} = -1$$

The required asymptotes from (ii) are respectively

$$y = x,\ y = -\frac{1}{2}x - \frac{1}{2} \text{ and } y = -2x - 1$$

or $\quad y = x,\ x + 2y + 1 = 0 \text{ and } 2x + y + 1 = 0$

Example 5. Find all asymptotes of the curve

$$4x^3 - 3xy^2 - y^3 + 2x^2 - xy - y^2 - 1 = 0 \qquad \text{(NEHU 2014)}$$

Solution: Let $y = mx + c$ ....(i) be an asymptote to the given curve

Putting $y = mx + c$ in the equation of the given curve we get

$4x^3 - 3x(mx + c)^2 - (mx + c)^3 + 2x^2 - x(mx + c) - (mx + c)^2 - 1 = 0$

or $4x^3 - 3m^2x^3 - 3c^2x - 6mcx^2 - m^3x^3 - c^3 - 3m^2cx^2 - 3mc^2x$
$\quad + 2x^2 - mx^2 - cx - m^2x^2 - 2mcx - c^2 - 1 = 0$

or $x^3(4 - 3m^2 - m^3) + x^2(-6mc - 3m^2c + 2 - m - m^2)$
$\quad + x(-3c^2 - 3mc^2 - c - 2mc) + (-c^3 - c^2 - 1) = 0$ ....(ii)

Equating the coefficients of x^3 and x^2 to zero we get

$4 - 3m^2 - m^3 = 0$ and $-6mc - 3m^2c + 2 - m - m^2 = 0$

Now $4 - 3m^2 - m^3 = 0 \Rightarrow (1 - m)(m + 2)^2 = 0$

$$\Rightarrow m = 1, -2, -2$$

Also $-6mc - 3m^2c + 2 - m - m^2 = 0$

$\Rightarrow \quad c(6m + 3m^2) = 2 - m - m^2$

$$\Rightarrow \quad c = \frac{2 - m - m^2}{6m + 3m^2} \text{ ....(iii)}$$

When m=1, from (iii) we have $c = \dfrac{2-1-1}{6+3} = 0$

When m=–2, from (iii) we have $c = \dfrac{2+2-4}{-12+12} = \dfrac{0}{0}$ indeterminate

Hence equating to zero the coefficient of x from (ii) we get

$$3c^2 - 3mc^2 - c - 2mc = 0$$

$\Rightarrow \quad 3c^2 + 3c = 0$ as $m = 1$

When $m = -2$ we get $3c(c + 1) = 0 \Rightarrow c = 0, -1$

The required asymptotes from (i) are respectively

$\therefore\ y = x;\ y = -2x - 1,\ y = -2x$

i.e $y = x,\ y + 2x + 1 = 0,\ y + 2x = 0$

Example 6. Find the asymptotes of the curve

$$x^2y^2 - a^2x^2 - a^2y^3 = 0$$

Solution: Equating the highest power of x i.e x^2 to zero we have $y^2 - a^2 = 0$ or $y = \pm a$ is the asymptotes parallel to x-axis

Since the coefficient of the highest power of y is a constant, so there is no asymptpte parallel to y-axis

Example 7. Find all asymptotes to the curve

$$y^2 (x^2 - a^2) = x$$

Solution: Equating the coefficient of the highest power of x i.e x^2 to zero we get $y^2 = 0$ i.e $y = 0$ i.e x axis itself is the asymptote to the given curve

Similarly equating the coefficient of the highest power of y i.e y^2 to zero we get $x^2 - a^2 = 0$ i.e $x = \pm a$ as the asymptotes parallel to y-axis of the curve

$\therefore$ The required asymptotes are $y=0$ and $x=\pm a$

Example 8. Find all asymptotes to the curve

$$\frac{a^3}{x^3} - \frac{b^3}{y^3} = 1$$

Solution: The given curve can be written as

$$a^3y^3 - b^3x^3 = x^3y^3$$

Equating the coefficients of highest powers of x and y i.e x^3 and y^3 in the above equation to zero we get $y^3 + b^3 = 0$ and $x^3 - a^3 = 0$

i.e $(y + b)(y^2 - by + b^2) = 0$ and $(x-a)(x^2 + ax + a^2) = 0$

Hence $y^3 + b^3 = 0$ gives one real asymptote $y+b=0$ and $x^3 - a^3 = 0$ gives one real asymptote $x=a$ and two each imaginary asymptotes

Since the curve being of six degree, there cannot be more than six asymptotes of which four are imaginary and two real asymptotes

$y = -b$ parallel to x-axis and

$x = a$ parallel to y-axis

Example 9. Find the asymptote to the curve $r\theta = \alpha$

Solution: The equation of the curve can be written as

$$\frac{1}{r} = \frac{1}{\alpha}\theta \quad \left(f(\theta) = \frac{1}{r}\right)$$

$$\therefore f(\theta) = \frac{1}{\alpha}\theta \Rightarrow f'(\theta) = \frac{1}{\alpha}$$

Equating $f(\theta)$ to zero we get $\theta\frac{1}{\alpha} = 0 \Rightarrow \theta = 0 = \beta$ (say)

$$\therefore \frac{1}{f'(\beta)} = \alpha$$

$\therefore$ The required asymptote is $r\sin(\theta - \alpha) = \frac{1}{f'(\beta)} = \alpha$

Example 10. Find the asymptote of the curve $r = \frac{a}{(1-\cos\theta)}$

Solution: The equation of the curve can be written as

$$\frac{1}{r} = \frac{1}{a}(1-\cos\theta) \quad \left(f(\theta) = \frac{1}{r}\right)$$

Here $f(\theta) = \frac{1}{a}(1-\cos\theta)$ and $f'(\theta) = \frac{1}{a}\sin\theta$

equating $f(\theta)$ to zero we get $1 - \cos\theta = 0$ or $\cos\theta = 1$ or $\theta = 2\kappa\pi,\ \kappa \in \mathbb{Z} = \alpha$ (say)

$$\therefore f'(\alpha) = \frac{1}{a}\sin 2\kappa\pi = 0$$

or $\frac{1}{f'(\alpha)} = \infty$ which being not finite

There is no asmptote to the given curve.

Exercise

1. Find the asymptotes of the following curves
   (i) $x^2y - xy^2 + xy + y^2 + x - y = 0$
   (ii) $x^2y^2 - x^2y - xy^2 + x + y + 1 = 0$
   (iii) $x^3 - 2y^3 + xy(2x-y) + y(x - y) + 1 = 0$
   (iv) $3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5y = 0$
   (v) $x^2(x-y)^2 - a^2(x^2 + y^2) = 0$
   (vi) $x^2y^2 - a^2x^2 - a^2y^3 = 0$
   (vii) $x^3 - 2x^2y + xy^2 - xy + x^2 + 2 = 0$ (NEHU 2013)

(viii) $y^2 - x^2 - 2x - 2y - 3 = 0$ (NEHU 2007)

(ix) $x^2y + xy^2 + xy + y^2 + 3x = 0$ (NEHU 2003)

(x) $x^2y = x^3 + x + y$

(xi) $y^3 - 6xy^2 + 11x^2y - 6x^3 + x + y = 0$

(xii) $y^3 - yx^2 + y^2 + x^2 - 4 = 0$ (NEHU 2004)

(xiii) $2x^3 - x^2y - 2xy^2 + y^3 - 4x^2 + 8xy - 4x + 1 = 0$ (NEHU 2008)

(xiv) $x^2 (x-y)^2 - a^2 (x^2 + y^2) = 0$

(xv) $y^3 - 6xy^2 + 11x^2y - 6x^3 + y^2 - x^2 + 2x - 3y - 1 = 0$

(xvi) $y^2 (x^2 - a^2) = x$

2. Find the asymptotes of the curve $x^2 (x^2+y^2-2xy)-2x^2-2y^2=0$ which are parallel to the line $x=y$

3. Show that the asymptotes of the curve

   $y^3 - xy^2 + 8x^2y - 4x^3 - 3y^2 + 9xy - 6x^2 + 2y - 2x = 1$ are $y = 1$, $y = 2x + 1$ and $y = 2x + 2$

4. Find the asymptotes of the following curves

   (i) $r = a (\cos\theta + \sec\theta)$ (ii) $r^n \sin n\theta = a^n$ $(n>1)$

   (iii) $r\cos\theta = 2a\sin\theta$ (iv) $r = a \operatorname{cosec}\theta + b$

   (v) $r = a\sec\theta + b\tan\theta$

5. Obtain the asymptotes of the curve $2x(y - 3)^2 - 3y (x - 1)^2 = 0$ and hence show that these asymptotes form a triangle and find the measure of its area

6. Show that the asymptotes of $x^2y^2 - a^2 (x^2 + y^2) - a^3 (x + y) + a^4 = 0$ form a square two of whose angular points lie on the curve.

14

Concavity, Convexity, Points of Inflexion

Introduction

In mathematics, we can have concave shape and convex shapes as well as concave and convex functions. In this lesson, through definition and example, we will learn what it means to be concave or convex and what these shapes and functions look like.

14.1 Definition

Concavity and Convexity

Let P be a point on the curve y = f(x) and PT be the tanget to the curve at P.

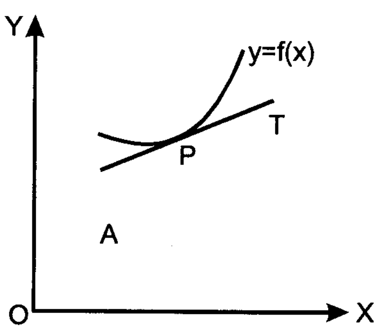


Fig. 1

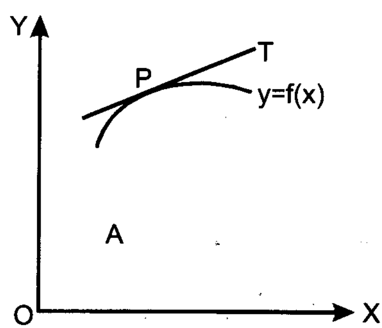


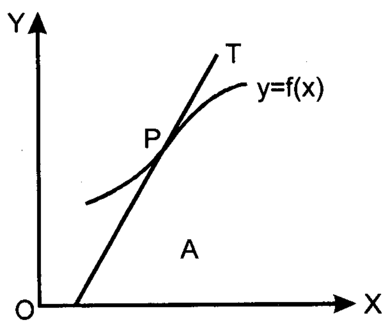
Fig. 2

Then the curve at P is said to be concave or convex with respect to a A (not lying on PT), according as a small portion of the curve in the immediate neighbourhood of P (on both sides of it) lies entirely on the same side of PT or an opposite side of PT with respect to A.

Note: A curve at the point P on it is is convex or concave with respect to a given line according as it is convex or concave with respect to the foot of the perpendicular from P on the line.

14.2 Point of Inflexion

If the curve crosses the tangent at P then we say that P is a point of inflexion on the curve. At this point clearly the curve, on one side of P, is convex and on the otherside it is concave with respect to any point A (not lying on the tangent line).



14.3 Test of Concavity or Convexity (with respect to x-axis)

Let P(x, y) be any point on the curve y = f(x), and Q (x+h, f(x+h)) be any neighbouring point of P (h, k being small positive or negative). Let PT be the tangent at P and let QM meets PT at R.

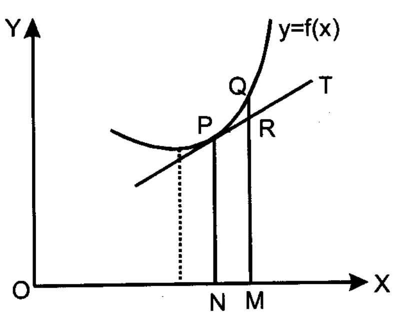


Fig. 1

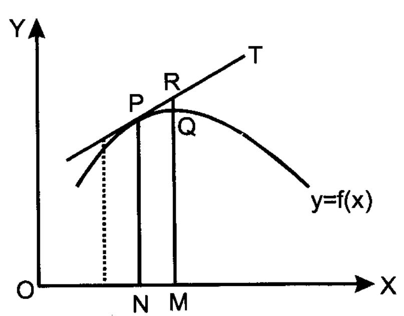


Fig. 2

Equation of the tangent PT is

$Y - y = f'(x)\,(X - x)$

Since NM = x + h $\quad \therefore$ RM = Y = y + hf′(x)

Also QM = f(x + h) = f(x) + hf′(x) + $\frac{h^2}{2!} f^2 (x + \theta h)$ $0<\theta<1$

$$QM - RM = \frac{h^2}{2!} f^2 (x + \theta h) \quad \text{...........(i)}$$

Assuming $f^2(x)$ to be continuous at P and $f^2(x) \neq 0$ hence $f^2 (x+\theta h)$ has the same sign as $f^2(x)$ when $|h|$ is sufficiently small.

$\therefore$ from (i) QM − RM has the same signs as that of $f^2(x)$ for $|h|$ being sufficiently small.

Hence from (i) QM > RM if $f^2(x) \left(\text{or } \frac{d^2y}{dx^2}\right)$ at P is positive, for Q on either side of P in its neighbourhood, and so the curve in the neighbourhood of P (an either side of it) is entirely above the tangent. Hence the curve at P is convex with respect to x-axis (fig 1)

Again from (i) QM < RM if $f^2(x) \left(\text{or } \frac{d^2y}{dx^2}\right)$ at P is negative, for Q on either side of P in its neighbourhood, and so the curve in the neighbourhood of P (an either side of it) is entirely below the tangent. Hence the curve at P is concave with respect to x-axis (fig 2)

Hence the criterion for convexity or concavity of a curve at a point with respect to x-axis are

(i) If y $\frac{d^2y}{dx^2} > 0$ at P, the curve at P is convex w.r.t x-axis

(ii) If y $\frac{d^2y}{dx^2} < 0$ at P, the curve at P is concave w.r.t x-axis

Note: At the point where the tangent is parallel to the y-axis, $\frac{dy}{dx}$ is infinite. At such point, instead of investigating with respect to x-axis, we investigate. Convexity or concavity of the curve with respect to y-axis and the Criterion will be similar to above as follows:

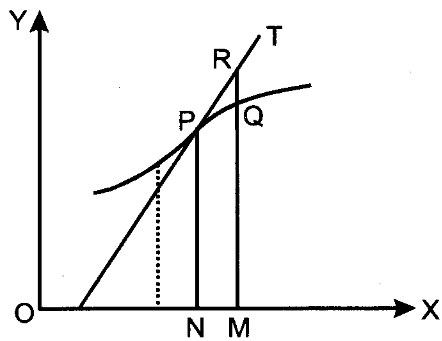
The curve at P is convex or concave with respect to y-axis according as

$x \frac{d^2y}{dy^2} > 0$ or < 0 at P

14.4 Condition for Point of Inflexion

Let $f^2(x) = 0$ at P and $f^3(x) \neq 0$ in the above investigation

Then $QM = f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f^2(x) + \frac{h^2}{3!} f^3(x + \theta h)$ $0<\theta<1$



$\therefore QM - RM = \frac{h^3}{3!} f^3(x + \theta h)$ $\qquad \because f^2(x) = 0$

As in previous discussion $f^3(x + \theta h)$ i.e QM–RM has the same sign as that of $\frac{h^3}{3!} f^3(x)$ for $|h|$ sufficiently small (which has opposite signs for positive and negative values of h whatever be the sign of $f^3(x)$ at P)

Thus, near P the curve is above the tangent on one side of P and below on the other side as in figure. Hence P is a point of inflexion.

Thus the condition that P is a point of inflexion on the curve y = f(x) is that

$$\frac{d^2y}{dx^2} = 0 \text{ and } \frac{d^3y}{dx^3} \neq 0 \text{ at P}$$

Note: At the point where the tangent is parallel to y axis, $\frac{dy}{dx}$ is infinte at P and the condition that P is a point of inflexion is that

$$\frac{d^2y}{dx^2} = 0 \text{ and } \frac{d^3y}{dx^3} \neq 0 \text{ at P}$$

Illustrative Examples

Examples 1. Examine convexity or concavity with respect to x-axis and determine its point of inflexion if any for the curve y = f(x) = sinx

Solution: Given curve y = sinx

$$\therefore \frac{dy}{dx} = \cos x \text{ and } \frac{d^2y}{dx^2} = -\sin x$$

Hence $y\frac{d^2y}{dx^2} = -\sin x$ which is negative for all value of x except those for which sinx = 0 i.e $x = \kappa\pi,\ \kappa \in \mathbb{Z}$

Thus the curve is concave to the x-axis at every point excepting at points where it crosses x-axis

$$\text{at } x = \kappa\pi,\ \frac{d^2y}{dx^2} = 0 \text{ and } \frac{d^3y}{dx^3} = -\cos x \neq 0$$

Hence these points are the points of inflexion.

Example 2. Show that the curve $y^3 = 8x^2$ is concave to the foot of the ordinate everywhere except at the origin. (NEHU 2013)

Solution: The given curve is $y^3 = 8x^2$ or $y = 2x^{2/3}$

$$\therefore \frac{dy}{dx} = \frac{4}{3}x^{-1/3} \text{ and } \frac{d^2y}{dx^2} = -\frac{4}{9}x^{-4/3}$$

$$\therefore y\frac{d^2y}{dx^2} = 2x^{2/3}\left(-\frac{4}{9}x^{-4/3}\right) = -\frac{8}{9}x^{-2/3} = -\frac{8}{9x^{2/3}}$$

Thus excepting at the origin $x^{2/3}$ being positive for all values of x, we see that $y\frac{d^2y}{dx^2} < 0$.

Hence the curve is concave everywhere to the foot of the ordinate except at the origin.

Example 3. Show that the curve y = logx is everywhere convex upwards but the curve y = xlogx is everywhere concave upwards

Solution: for the curve y = logx we have

$$\frac{dy}{dx} = \frac{1}{x} \text{ and } \frac{d^2y}{dx^2} = -\frac{1}{x^2}$$

which always remains negative at all points.

Hence the curve y=logx is convex upwards (or concave downwards) at all points.

For the curve y = xlogx we have

$$\frac{dy}{dx} = 1 + \log x \text{ and } \frac{d^2y}{dx^2} = \frac{1}{x}$$

which is always positive for positive values of x (note that x cannot be negative)

Hence the curve y = xlogx is everywhere concave upwards

Example 4. Find the points of inflexion if any, for the curve $x = (\log y)^3$

(NEHU 2017)

Solution: The curve is $x = (\log y)^3$

$$\therefore \frac{dx}{dy} = \frac{3(\log y)^2}{y} \text{ and } \frac{d^2x}{dy^2} = \frac{3\log y}{y^2}(2-\log y)$$

Also $\frac{d^3x}{dy^3} = \frac{6(\log y)^3 - 18(\log y) + 6}{y^3}$

Now if $\frac{d^2x}{dy^2} = 0$ Then $\frac{3\log y}{y^2}(2-\log y) = 0$

i.e $\log y = 0$ or $2 = \log y$

i.e $y=1$ or $y = e^2$

$\therefore \frac{d^2x}{dy^2} = 0$ at $y = 1$ and at $y = e^2$

where $\frac{d^3x}{dy^3} \neq 0$

Hence if $y = 1$ $x = 0$ and if $y = e^2$, $x = 8$

$\therefore$ The points of inflexion are $(0, 1)$ and $(8, e^2)$

Exercises

1. Show that $y = x^4$ is concave upwards at the origin and $y = e^x$ is everywhere concave upwards.
2. Prove that $(a-2, -2/e^2)$ is a point of inflexion of the curve $y=(x-a)e^{x-a}$
3. Find the point of inflexion if any of the curve $y = 3 + 6(x-)^5$
4. Show that the curve $y = e^{-x^2}$ has inflexions at $x = \pm\frac{1}{\sqrt{2}}$
5. Find the points of inflexion if any on the curve $c^2y = (x-a)^3$
6. Show that the curve $(y-a)^3 = a^3 - 2a^2x + ax^2$ where $a>0$ is always concave the x-axis
7. Show that the points of inflexion of the curve $y^2 = (x-a)^2(x-b)$ lie on the line $3x + a = 4b$

15

Partial Differentiation

Introduction

When a function depends on more than one variable, we can use the partial derivative to determine how that function changes with respect to one variable at a time. In this lesson we use examples to define partial derivatives and to explain the rules for evaluating them.

15.1 Definition

Let u be a symbol which has a definite value for every pair of values x and y. Then u is called a function of two independent variables x and y and is written as u = f(x, y)

15.2 Partial Differential Coefficients

The partial differential coefficient of f(x, y) with respect to x is defined as $\lim_{\delta x \to 0} \frac{f(x+\delta x, y)-f(x,y)}{\delta x}$ provided this limit exists and is written as $\frac{\partial f}{\partial x}$ or f_x or $D_x f$.

Thus the partial differential coefficient of f(x, y) with respect x is the ordinary differential coefficient of f(x, y) when y is regarded as a constant.

Similarly, the partial differential coefficient of f(x, y) with respect to y is defined as $\lim_{\delta y \to 0} \frac{f(x, y+\delta y)-f(x,y)}{\delta y}$ provided thus limit exists and is written as $\frac{\partial f}{\partial y}$ or f_y or D_y f.

Thus, the partial differential coefficient of f(x, y) with respect to y is the ordinary differential coefficient of f(x, y) when x is regarded as a constant.

We can further differentiate f_x and f_y partially with respect to x and y

Then partial differential coefficient of $\frac{\partial f}{\partial x}$ with respect to x and y are respectively $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^2 f}{\partial x \partial y}$ or f_{xx} and f_{xy}.

Similarly, the partial differential coefficient of $\frac{\partial f}{\partial x}$ with respect to x and y are respectively $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial y^2}$ or f_{yx} and f_{yy}.

Thus the usual notations for the second order partial derivatives of u = f(x, y) are as follows

(i) $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)$ i.e $\frac{\partial^2 u}{\partial x^2}$ or f_{xx}

(ii) $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)$ i.e $\frac{\partial^2 u}{\partial y^2}$ or f_{yy}

(iii) $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$ i.e $\frac{\partial^2 u}{\partial x \partial y}$ or f_{xy}

(iv) $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$ i.e $\frac{\partial^2 u}{\partial y \partial x}$ or f_{yx}

In all ordinatry cases $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ i.e $f_{xy} = f_{yx}$

15.3 Remarks

(i) $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} f(x, y) = f_x(x, y)$

$= \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$ provided this limit exists

(ii) $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y} f(x, y) = f_y(x, y)$

$$= \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x, y)}{\delta y} \text{ provided this limit exists}$$

(iii) $$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}(f_x(x, y) = f_{xx}(x, y)$$

$$= \lim_{\delta x \to 0} \frac{f_x(x+\delta x, y) - f_x(x, y)}{\delta x} \text{ provided this limit exists}$$

(iv) $$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}(f_y(x, y) = f_{yy}(x, y)$$

$$= \lim_{\delta y \to 0} \frac{f_y(x, y+\delta y) - f_y(x, y)}{\delta y} \text{ provided this limit exists}$$

(v) $$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\{(f_y(x, y)\} = f_{xy}(x, y)$$

$$= \lim_{\delta x \to 0} \frac{f_y(x+\delta x, y) - f_y(x, y)}{\delta x} \text{ provided this limit exists}$$

(vi) $$\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial y}\{(f_x(x, y)\} = f_{yx}(x, y)$$

$$= \lim_{\delta y \to 0} \frac{f_x(x, y+\delta y) - f_x(x, y)}{\delta y} \text{ provided this limit exists}$$

15.4 Total Differential Coefficient

If u = f(x, y) where x and y are functions of a third variable 't' connected by the relation $x = \phi(t)$ and $y = \psi(t)$, then $\frac{du}{dt}$ is called the total differential coefficient of u.

We can find the value of $\frac{du}{dt}$ by substituting the values of x and y in terms of t in the value of u and then finding the simple derivative of u with respect to t.

15.5 To find $\frac{du}{dt}$ without actually substituting the values of x and y in u = f(x, y)

Let t be changed to $t+\delta t$ and let the corresponding changes in u, x, y be δu, δx and δy respectively. Therefore, we have

$$u = f(x, y)$$

and $u + \delta u = f(x + \delta x, y + \delta y)$

$\therefore \quad \delta u = f(x + \delta x, y + \delta y) - u$

$\qquad = f(x + \delta x, y + \delta y) - f(x, y)$

or $\quad \delta u = \{f(x + \delta x, y + \delta y) - f(x, y + \delta y)\}$
$\qquad + \{f(x, y + \delta y) - f(x, y)\}$

$$\therefore \quad \frac{\delta u}{\delta t} = \left\{\frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta t}\right\} + \left\{\frac{f(x, y+\delta y) - f(x, y)}{\delta t}\right\}$$

$$\text{or} \quad \frac{\delta u}{\delta t} = \left\{\frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x}\right\}.\frac{\delta x}{\delta t} + \left\{\frac{f(x, y+\delta y) - f(x, y)}{\delta y}\right\}.\frac{\delta y}{\delta t}$$

Proceeding to the limit as $\delta t \to 0$ consequently $\delta x \to 0$ and $\delta y \to 0$ also. Hence we have

$$\frac{du}{dt} = \lim_{\delta x \to 0}\left\{\frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x}\right\} \lim_{\delta t \to 0}\frac{\delta x}{\delta t}$$

$$+ \lim_{\delta y \to 0}\left\{\frac{f(x, y+\delta y) - f(x, y)}{\delta y}\right\} \lim_{\delta t \to 0}\frac{\delta y}{\delta t} \quad ....(i)$$

$$\text{Now} \lim_{\delta x \to 0}\frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} = f_x(x, y + \delta y)$$

$$= \frac{\partial}{\partial x} f(x, y + \delta y)$$

$$\text{and} \lim_{\delta y \to 0}\frac{\partial}{\partial x} f(x, y+\delta y) = \frac{\partial}{\partial x} f(x, y)$$

Hence we shall assume that

$$\lim_{\delta x \to 0}\frac{f(x+\delta x, y+\delta y) - f(x, y+\delta y)}{\delta x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial f}{\partial x}$$

Hence from (i) we have

$$\frac{du}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt} \quad \text{......... (ii)}$$

or $$u_x\frac{dx}{dt}+u_y\frac{dy}{dt}=0$$

In particular if u is a function of x, y and y is a function of x, then by putting t=x in (ii) we have

$$\frac{du}{dt}=\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dt} \quad \text{............(iii)}$$

Thus in general if u= $f(x_1, x_2, x_3, ...x_n)$ and $x_1, x_2, x_3, ...x_n$ are all functions of t, then similarly

$$\frac{du}{dt}=\frac{\partial u}{\partial x_1}\frac{dx_1}{dt}+\frac{\partial u}{\partial x_2}\frac{dx_2}{dt}+......+\frac{\partial u}{\partial x_n}\frac{dx_n}{dt}$$

15.6 Important Case

If f(x, y) = c, a constant, then from (ii) above we have

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0$$

or $$\frac{dy}{dx}=-\frac{\partial f/\partial x}{\partial f/\partial y} \text{ provided } \frac{\partial f}{\partial y}\neq 0$$

or $$\frac{\partial y}{\partial x}=-\frac{f_x}{f_y},\ f_y\neq 0$$

15.7 Change of Variables

If u = f(x, y) where u = $\phi(t_1, t_2)$, y = $\psi(t_1, t_2)$ then as in 15.5 above we have

$$\frac{\partial u}{\partial t_1}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial t_1}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t_1} \text{ and } \frac{\partial u}{\partial t_2}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial t_2}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial t_2} \quad \text{......(i)}$$

If the values of t_1 and t_2 can be easily obtained in terms of x, y say $t_1 = F_1(x, y)$ and $t_2 = F_2(x, y)$ then

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial t_1}\frac{\partial t_1}{\partial x}+\frac{\partial u}{\partial t_2}\frac{\partial t_2}{\partial x} \text{ and } \frac{\partial u}{\partial y}=\frac{\partial u}{\partial t_1}\frac{\partial t_1}{\partial y}+\frac{\partial u}{\partial t_2}\frac{\partial t_2}{\partial y} \quad \text{.....(ii)}$$

15.8 Homegeneous Functions

The expression $a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + ...+a_ny^n$ in x and y of degree n is called a homogeneous function of x and y of degree n.

$$f(x, y) = a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + ...+a_ny^n$$

$$= x_n\left[a_0 + a_1\left(\frac{y}{x}\right) + a_2\left(\frac{y}{x}\right)^2 + ...+ a_n\left(\frac{y}{x}\right)^n\right]$$

$$= x^n \phi\left(\frac{y}{x}\right)$$

Hence every homogeneous function of x and y of degree n can be written as $x^n \phi\left(\frac{y}{x}\right)$

15.9 Euler's Terorem on Homogeneous Functions

Statements: If f(x, y) is a homogeneous function of x and y of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$$

Proof: Since f(x, y) is a homogeneous function in x and y of degree n

then, $f(x, y) = x^n \phi\left(\frac{y}{x}\right)$

$$\therefore \quad \frac{\partial f}{\partial x} = nx^{n-1} \phi\left(\frac{y}{x}\right) + x^n \phi'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

$$\therefore \quad x\frac{\partial f}{\partial x} = nx^n \phi\left(\frac{y}{x}\right) - yx^{n-1} \phi'\left(\frac{y}{x}\right) \quad ....(i)$$

and $$\frac{\partial f}{\partial y} = x^n \phi'\left(\frac{y}{x}\right) \cdot \frac{1}{x} = x^{n-1} \phi'\left(\frac{y}{x}\right)$$

$$\therefore \quad y\frac{\partial f}{\partial y} = yx^{n-1} \phi'\left(\frac{y}{x}\right) \quad ....(ii)$$

Adding (i) and (ii) we get

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nx^n \phi\left(\frac{y}{x}\right) - yx^{n-1}\phi'\left(\frac{y}{x}\right) + yx^{n-1}\phi'\left(\frac{y}{x}\right)$$

or $$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nx^n \phi\left(\frac{y}{x}\right)$$

or $$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y) = nf \text{ Proved}$$

Cor: If u is a homogeneous function of x and y of degree n, then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$ (NEHU 2004)

Proof: Since u is a homogeneous function of x and y of degree n then by 15.9 above

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu \quad \text{........ (i)}$$

Differentiating (i) partially with respect to x we have

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = n\frac{\partial u}{\partial x} \quad \text{.........(ii)}$$

Differentiating (i) partially with respect to y we have

$$x\frac{\partial^2 u}{\partial y\partial x} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial y^2} = n\frac{\partial u}{\partial y} \quad \text{.........(iii)}$$

Multiplying (ii) by x and (iii) by y and adding we get

$$\left(x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right) + \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = n\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

or $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(nu) - nu \text{ by (i)}$$

or $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u \text{ Proved}$$

Illustrative Examples

Example 1. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ when

(i) $f = \sin^{-1}\left(\frac{y}{x}\right)$ (ii) $f = x^y$ (iii) $f = ye^{-x/y}$

(iv) $f = x \tan y + y \tan x$

Solution: (i) Given $f = \sin^{-1}\left(\frac{y}{x}\right)$

$$\therefore \quad \frac{\partial f}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x\sqrt{x^2-y^2}}$$

$$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{y}{x}\right)^2}}\left(\frac{1}{x}\right) = \frac{1}{\sqrt{x^2-y^2}}$$

(ii) Given $f = x^y \Rightarrow \log f = y \log x$ ....(i)

Differentiating (i) partially w.r.t x we get

$$\frac{1}{f}\frac{\partial f}{\partial x} = \frac{y}{x}$$

or $$\frac{\partial f}{\partial x} = f\frac{y}{x} = x^y.\frac{y}{x} = x^{y-1}.y$$

Again differentiating (i) paratially w.r.t y we get

$$\frac{1}{f}\frac{\partial f}{\partial y} = 1.\log x \text{ or } \frac{\partial f}{\partial y} = f\log x$$

$$\text{or } \frac{\partial f}{\partial y} = x^y \log x$$

(iii) Given $f = ye^{-x/y}$

$$\therefore \quad \frac{\partial f}{\partial x} = ye^{-x/y}\left(-\frac{1}{y}\right) = -e^{-x/y}$$

$$\frac{\partial f}{\partial y} = ye^{-x/y}\left(-\frac{x}{y^2}\right) + e^{-x/y}$$

or $$\frac{\partial f}{\partial y} = e^{-x/y}\left[\frac{x}{y}+1\right]$$

or $$\frac{\partial f}{\partial y} = \left(\frac{x+y}{y}\right)e^{-x/y}$$

(iv) Given f = x tany + y tanx

$\therefore$ $$\frac{\partial f}{\partial x} = 1.\tan y + y\sec^2 x = \tan y + y\sec^2 x$$

$$\frac{\partial f}{\partial x} = x\sec^2 y + 1\tan x = x\sec^2 y + \tan x$$

Example 2. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ for the following functions

(i) $\tan^{-1}\left(\frac{x}{y}\right)$ (ii) $\log(x^2 + y^2)$

Solution: (i) Let $f = \tan^{-1}\left(\frac{x}{y}\right)$

$\therefore$ $$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2}$$

and $$\frac{\partial f}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x} = \frac{x}{x^2+y^2}$$

(ii) Let $f = \log(x^2 + y^2)$

$\therefore$ $$\frac{\partial f}{\partial x} = \frac{1}{x^2+y^2}.2x = \frac{2x}{x^2+y^2}$$

$$\frac{\partial f}{\partial x} = \frac{1}{x^2+y^2}.2y = \frac{2y}{x^2+y^2}$$

Example 3. Verify that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ when

(i) $z = \log\left\{\frac{(x^2+y^2)}{xy}\right\}$ (ii) $z = \log \tan\left(\frac{y}{x}\right)$

Solution: (i) $z = \log\left\{\frac{(x^2+y^2)}{xy}\right\}$ ....(i)

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{1}{\frac{x^2+y^2}{xy}}\left[\frac{xy2x-(x^2+y^2).y}{x^2y^2}\right]$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = \frac{xy}{x^2+y^2}\frac{yx^2-y^3}{x^2y^2} = \frac{y(x^2-y^2)}{xy(x^2+y^2)} = \frac{x^2-y^2}{x(x^2+y^2)}$$

$$\therefore \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left[\frac{x^2-y^2}{x(x^2+y^2)}\right]$$

$$= \frac{1}{x}\left[\frac{(x^2+y^2)(-2y)-(x^2-y^2).2y}{(x^2+y^2)^2}\right]$$

$$\Rightarrow \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{1}{x}\left[\frac{-2x^2y-2y^3-2yx^2+2y^3}{(x^2+y^2)^2}\right]$$

$$\text{or} \quad \frac{\partial^2 z}{\partial y \partial x} = \frac{1}{x}\left(\frac{-4x^2y}{(x^2+y^2)^2}\right) = -\frac{4yx}{(x^2+y^2)^2} \quad \text{........(ii)}$$

Again from (i) $\frac{\partial}{\partial y} = \frac{1}{\frac{x^2+y^2}{xy}}\left[\frac{xy(2y)-(x^2+y^2).x}{x^2y^2}\right]$

$$\text{or} \quad \frac{\partial}{\partial y} = \frac{xy}{x^2+y^2}\left[\frac{2xy^2-x^3-y^2x}{x^2y^2}\right]$$

$$= \frac{xy}{x^2+y^2} \frac{xy^2-x^3}{x^2y^2}$$

$$= \frac{y^2-x^2}{y(x^2+y^2)}$$

$$\therefore \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{y^2-x^2}{y(x^2+y^2)}\right]$$

$$= \frac{1}{y}\left[\frac{\left(x^2+y^2\right)(-2x)-\left(y^2-x^2\right).2x}{\left(x^2+y^2\right)^2}\right]$$

$$\text{or} \quad \frac{\partial^2 z}{\partial x\,\partial y} = \frac{1}{y}\left[\frac{-2x^3-2y^2x-2xy^2+2x^3}{\left(x^2+y^2\right)^2}\right]$$

$$= \frac{1}{y}\frac{-4xy^2}{\left(x^2+y^2\right)^2} = -\frac{4xy}{\left(x^2+y^2\right)^2} \quad \text{..........(iii)}$$

By (ii) and (iii) we have

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$$

(ii) Given $z = \log \tan\left(\frac{y}{x}\right)$ ....(i)

$$\therefore \quad \frac{\partial z}{\partial x} = \frac{1}{\tan\left(\frac{y}{x}\right)} \sec^2\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

$$= -\frac{y}{x^2}\frac{1}{\sin\left(\frac{y}{x}\right)\cos\left(\frac{y}{x}\right)}$$

$$\text{or} \quad \frac{\partial z}{\partial x} = -\frac{2y}{x^2.2\sin\left(\frac{y}{x}\right)\cos\left(\frac{y}{x}\right)} = -\frac{2y}{x^2\sin\left(\frac{2y}{x}\right)}$$

$$\therefore \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left[-\frac{2y}{x^2 \sin\left(\frac{2y}{x}\right)}\right]$$

$$= -\frac{2}{x^2}\left[\frac{\sin\left(\frac{2y}{x}\right) - y\cos\left(\frac{2y}{x}\right)\left(\frac{2}{x}\right)}{\sin^2\left(\frac{2y}{x}\right)}\right]$$

or $$\frac{\partial^2 z}{\partial y \partial x} = -\frac{2}{x^2}\left[\frac{1}{\sin\left(\frac{2y}{x}\right)} - \frac{2y}{x}\frac{\cos\left(\frac{2y}{x}\right)}{\sin\left(\frac{2y}{x}\right)} \cdot \frac{1}{\sin\left(\frac{2y}{x}\right)}\right]$$

$$= -\frac{2}{x^2}\left[\operatorname{cosec}\left(\frac{2y}{x}\right) - \frac{2y}{x}\cot\left(\frac{2y}{x}\right)\operatorname{cosec}\left(\frac{2y}{x}\right)\right]$$

$$= -\frac{2}{x^3}\operatorname{cosec}\left(\frac{2y}{x}\right)\left[x - 2y\cot\left(\frac{2y}{x}\right)\right]$$

Again from (i) $$\frac{\partial z}{dy} = \frac{1}{\tan\left(\frac{y}{x}\right)}\sec^2\left(\frac{y}{x}\right).\frac{1}{x}$$

$$= \frac{1}{\sin\left(\frac{y}{x}\right)\cos\left(\frac{y}{x}\right)}.\frac{1}{x}$$

or $$\frac{\partial z}{dy} = \frac{2}{2\sin\left(\frac{y}{x}\right)\cos\left(\frac{y}{x}\right)}.\frac{1}{x}$$

$$= \frac{2}{x\sin\left(\frac{2y}{x}\right)}$$

$$\therefore \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left[\frac{2}{x\sin\left(\frac{2y}{x}\right)}\right]$$

$$= \frac{-2\left[\sin\left(\frac{2y}{x}\right) + x.\cos\left(\frac{2y}{x}\right)\left(-\frac{2y}{x^2}\right)\right]}{x^2 \sin^2\left(\frac{2y}{x}\right)}$$

or $$\frac{\partial^2 z}{\partial x\, \partial y} = -\frac{2}{x^2}\left[\frac{1}{\sin\left(\frac{2y}{x}\right)} - \frac{2y}{x}\frac{\cos\left(\frac{2y}{x}\right)}{\sin^2\left(\frac{2y}{x}\right)}\right]$$

$$= -\frac{2}{x^2}\left[\operatorname{cosec}\left(\frac{2y}{x}\right) - \frac{2y}{x}\cot\left(\frac{2y}{x}\right)\operatorname{cosec}\left(\frac{2y}{x}\right)\right]$$

or $$\frac{\partial^2 z}{\partial x\, \partial y} = -\frac{2}{x^2}\operatorname{cosec}\left(\frac{2y}{x}\right)\left[1 - \frac{2y}{x}\cot\left(\frac{2y}{x}\right)\right]$$

or $$\frac{\partial^2 z}{\partial x\, \partial y} = -\frac{2}{x^3}\operatorname{cosec}\left(\frac{2y}{x}\right)\left[x - 2y\cot\left(\frac{2y}{x}\right)\right] \quad ......(iii)$$

By (ii) and (ii) we have

$$\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial x\, \partial y}$$

Example 4. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$$

Solution: Given $u = \log(x^3 + y^3 + z^3 - 3xyz)$

$$\therefore \quad \frac{\partial u}{\partial y} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz}$$

$$\therefore \quad \frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3(x^2-yz+y^2-zx+z^2-xy)}{x^3+y^3+z^3-3xyz}$$

$$=\frac{3(x^2+y^2+z^2-xy-yz-zx)}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

or $$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{3}{x+y+z}$$

Example 5. If $u = \log(x^2 + y^2 + z^2)$; show that

$$x\frac{\partial^2 u}{\partial y\partial z}=y\frac{\partial^2 u}{\partial z\partial x}=z\frac{\partial^2 u}{\partial x\partial y}$$ (NEHU 2014)

Solution: Given $u = \log(x^2 + y^2 + z^2)$

$$\therefore \quad \frac{\partial u}{\partial x}=\frac{2x}{x^2+y^2+z^2}$$

$$\frac{\partial u}{\partial y}=\frac{2y}{x^2+y^2+z^2}$$

$$\frac{\partial u}{\partial z}=\frac{2z}{x^2+y^2+z^2}$$

$$\therefore \quad \frac{\partial^2 u}{\partial y\partial z}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right)=\frac{\partial}{\partial y}\left(\frac{2z}{x^2+y^2+z^2}\right)$$

$$=2z\frac{0-1(2y)}{\left(x^2+y^2+z^2\right)^2}$$

$$=-\frac{4yz}{\left(x^2+y^2+z^2\right)^2}$$

$$\therefore \quad x\frac{\partial^2 u}{\partial y\partial z}=-\frac{4xy^2}{\left(x^2+y^2+z^2\right)^2} \quad .....(i)$$

$$\frac{\partial^2 u}{\partial z\partial x}=\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial z}\left(\frac{2x}{x^2+y^2+z^2}\right)$$

$$=2x\frac{0-1.(2z)}{\left(x^2+y^2+z^2\right)^2}$$

or $$\frac{\partial^2 u}{\partial z \partial x} = -\frac{4xz}{\left(x^2+y^2+z^2\right)^2}$$

$$\therefore \quad y\frac{\partial^2 u}{\partial z \partial x} = -\frac{4xyz}{\left(x^2+y^2+z^2\right)^2} \quad ....(ii)$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{2y}{x^2+y^2+z^2}\right)$$

$$= -\frac{4xy}{\left(x^2+y^2+z^2\right)^2} \quad \text{(as above)}$$

$$z\frac{\partial^2 u}{\partial x \partial y} = -\frac{4xyz}{\left(x^2+y^2+z^2\right)^2} \quad .... (iii)$$

By (i), (ii) and (iii) we get

$$x\frac{\partial^2 u}{\partial y \partial z} = y\frac{\partial^2 u}{\partial z \partial x} = z\frac{\partial^2 u}{\partial x \partial y}$$

Example 6. If $u = f\left(\frac{y}{x}\right)$ show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

Solution: Given $u = f\left(\frac{y}{x}\right)$

$$\therefore \quad \frac{\partial u}{\partial x} = f'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

$$\therefore \quad x\frac{\partial u}{\partial x} = -\frac{y}{x} f'\left(\frac{y}{x}\right)$$

Also $$\frac{\partial u}{\partial y} = f'\left(\frac{y}{x}\right)\left(1/x\right)$$

$$\therefore \quad y\frac{\partial u}{\partial y} = \frac{y}{x} f'\left(\frac{y}{x}\right)$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{y}{x}f'\left(\frac{y}{x}\right) + \frac{y}{x}f'\left(\frac{y}{x}\right) = 0$$

Example 7. If $y = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

Solution: Given $u = \text{Sin}^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$ ......(i)

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left(\frac{1}{y}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)$$

$$= \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

$$\therefore \quad x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} \quad \text{....(ii)}$$

Again from (i) $\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left(-\frac{x}{y^2}\right) + \frac{1}{1+\left(\frac{y}{x}\right)^2}$

$$= -\frac{x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

$$\therefore y\frac{\partial u}{\partial y} = \frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \quad \text{......... (iii)}$$

Adding (ii) and (iii) we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} - \frac{x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} = 0$$

Example 8. If $V = \left(x^2+y^2+z^2\right)^{-1/2}$ show that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Solution: Given $V = (x^2+y^2+z^2)^{-1/2}$ ....(i)

$$\therefore \quad \frac{\partial V}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-1/2-1}(2x)$$

$$= \frac{-x}{(x^2+y^2+z^2)^{3/2}} = -x(x^2+y^2+z^2)^{-3/2}$$

$$\therefore \quad \frac{\partial^2 V}{\partial x^2} = -(x^2+y^2+z^2)^{-3/2} - x\left(-\frac{3}{2}\right)(x^2+y^2+z^2)^{-3/2-1}2x$$

$$= -(x^2+y^2+z^2)^{-3/2} + 3x^2(x^2+y^2+z^2)^{-5/2}$$

Illy from (i) we get

$$\frac{\partial^2 V}{\partial y^2} = -(x^2+y^2+z^2)^{-3/2} + 3y^2(x^2+y^2+z^2)^{-5/2}$$

$$\frac{\partial^2 V}{\partial z^2} = -(x^2+y^2+z^2)^{-3/2} + 3z^2(x^2+y^2+z^2)^{-5/2}$$

$$\therefore \quad \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2} = -3(x^2+y^2+z^2)^{-3/2} + (x^2+y^2+z^2)^{-5/2}(3x^2+3y^2+3z^2)$$

$$= -3(x^2+y^2+z^2)^{-3/2} + 3(x^2+y^2+z^2)^{-3/2}$$

$$= 0$$

Example 9. If $V = (x^2+y^2+z^2)^{-1/2}$ show that

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = -V$$

Solution: Given $V = (x^2+y^2+z^2)^{-1/2}$

$$\therefore \quad \frac{\partial V}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-1/2-1}(2x)$$

$$= -x(x^2+y^2+z^2)^{-3/2}$$

$$\therefore\ x\frac{\partial V}{\partial x} = -\ x^2\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\text{Similarly } y\ \frac{\partial V}{\partial y} = -\ y^2\left(x^2+y^2+z^2\right)^{-3/2}$$

$$z\ \frac{\partial V}{\partial z} = -\ z^2\left(x^2+y^2+z^2\right)^{-3/2}$$

$$\therefore\quad x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}+z\frac{\partial V}{\partial z} = \left(x^2+y^2+z^2\right)^{-3/2}\ (-x^2-y^2-z^2)$$

$$= -\left(x^2+y^2+z^2\right)^{-3/2}\ (x^2+y^2+z^2)$$

$$= -\left(x^2+y^2+z^2\right)^{-1/2} = -\ V$$

Example 10. Verify Euler's Theorem for the function $f(x, y) = ax^2 + 2hxy + by^2$

Solution: Here $f(x, y) = ax^2 + 2hxy + by^2$

$$= x^2\left[b\left(\frac{y}{x}\right)^2+2h\left(\frac{y}{x}\right)+a\right]$$

$$= x^2\ \phi\left(\frac{y}{x}\right)$$

Hence f(x, y) is a homogeneous function of degree 2 in x and y. Hence we are to prove that

$$x\frac{\partial f}{\partial x}+y\ \frac{\partial f}{\partial y} = 2f$$

Now $f = ax^2 + 2hxy + by^2$

$$\therefore\quad \frac{\partial f}{\partial x} = 2ax + 2hy \Rightarrow x\ \frac{\partial f}{\partial x} = 2ax^2 + 2hxy$$

$$\frac{\partial f}{\partial y} = 2hx + 2by \Rightarrow y\ \frac{\partial f}{\partial y} = 2hxy + 2by^2$$

$$\therefore\quad x\ \frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2ax^2 + 4hxy + 2by^2$$

$$= 2(ax^2 + 2hxy + by^2)$$

$$= 2f$$

which verfies Euler's Theorem.

Example 11. If $u = \sin^{-1}\dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$, show by Euler's Theorem that

$$\frac{\partial u}{\partial x}=-\frac{y}{x}\frac{\partial u}{\partial y} \text{ or } x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$$

Solution: Given $\text{Sinu} = \dfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}$

or $\text{Sinu} = \dfrac{\sqrt{x}\left[1-\sqrt{y/x}\right]}{\sqrt{x}\left[1+\sqrt{y/x}\right]}$

or $\text{Sinu} = x^0\,\phi\left(\dfrac{y}{x}\right)$

Hence Sinu is a homogeneous function of x and y of degree 0.

∴ By Euler's theorem,

$$x\frac{\partial}{\partial x}\text{Sinu}+y\frac{\partial}{\partial y}\text{Sinu}=0.\text{Sinu}$$

or $x\dfrac{\partial}{\partial u}\text{Sinu}\,\dfrac{\partial u}{\partial x}+y\dfrac{\partial}{\partial u}\text{Sinu}\dfrac{\partial u}{\partial y}=0$

or $x\,\text{Cosu}\,\dfrac{\partial u}{\partial x}+y\,\text{Cosu}\dfrac{\partial u}{\partial y}=0$

or $x\,\dfrac{\partial u}{\partial x}+y\,\dfrac{\partial u}{\partial y}=0$

Exmple 12. If $u = \sin^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}=\dfrac{1}{2}\tan u$

Solution: Given $\text{Sinu} = \left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right) = \dfrac{x\left[1+\left(\frac{y}{x}\right)\right]}{\sqrt{x}\left[1+\left(\frac{y}{x}\right)\right]}$

or $\quad \text{Sinu} = \sqrt{x}\left[\dfrac{1+\left(\dfrac{y}{x}\right)}{1+\left(\dfrac{y}{x}\right)}\right]$

or $\quad \text{Sinu} = x^{1/2}\,\phi\left(\dfrac{y}{x}\right)$

Hence Sinu is a homogeneous function in x and y of degree $\frac{1}{2}$

By Euler's Theorem we have

$$x\frac{\partial}{\partial x}\text{Sinu} + y\frac{\partial}{\partial y}\text{Sinu} = \frac{1}{2}\text{Sinu}$$

or $$x\frac{\partial}{\partial u}\text{Sinu}\frac{\partial u}{\partial x} + y\frac{\partial}{\partial u}\text{Sinu}\frac{\partial u}{\partial y} = \frac{1}{2}\text{Sinu}$$

or $$x\,\text{Cosu}\frac{\partial u}{\partial x} + y\ \text{Cosu}\frac{\partial u}{\partial y} = \frac{1}{2}\text{Sinu}$$

or $$x\,\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$$

Example 13. If $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x-y}\right)$, prove that

$$x\,\frac{\partial u}{\partial x} + y\,\frac{\partial u}{\partial y} = \sin 2u$$ (NEHU 2007)

Solution: Given $\tan u = \left(\dfrac{x^3+y^3}{x-y}\right) = \dfrac{x^3\left[1+\left(y/x\right)^3\right]}{x\left[1-\left(y/x\right)\right]}$

or $\tan u = x^2\,\phi\left(\dfrac{y}{x}\right)$

Hence tanu is a homogeneous function in x and y of degree 2

∴ By Euler's Theorem

$$x \frac{\partial}{\partial x} \tan u + y \frac{\partial}{\partial y} \tan u = 2 \tan u$$

i.e $$x \frac{\partial}{\partial u} \tan u \frac{\partial u}{\partial x} + y \frac{\partial}{\partial u} \tan u \frac{\partial u}{\partial y} = 2 \tan u$$

or $$x \sec^2 u \frac{\partial u}{\partial x} + y \sec^2 u \frac{\partial u}{\partial y} = 2 \tan u$$

or $$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \frac{\tan u}{\sec^2 u}$$

or $$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \frac{\sin u}{\cos u} \cos^2 u = 2 \sin u \cos u$$

or $$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sin 2u$$

Example 14. If $u = \cos^{-1}\left[\frac{x+y}{\sqrt{x}+\sqrt{y}}\right]$, show that

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} - \frac{1}{2} \cot u = 0$$

Solution: Same as Example 13.

Example 15. If $V = \log r$ and $r^2 = x^2 + y^2 + z^2$, then prove that

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{r^2}$$

Solution: Since $V = \log r$ and $r^2 = x^2 + y^2 + z^2$,

∴ $$V = \log\left(x^2 + y^2 + z^2\right)^{1/2}$$

∴ $$\frac{\partial V}{\partial x} = \frac{\frac{1}{2}\left(x^2 + y^2 + z^2\right)^{-1/2} (2x)}{\left(x^2 + y^2 + z^2\right)^{1/2}} = \frac{x}{x^2 + y^2 + z^2}$$

∴ $$\frac{\partial^2 V}{\partial x^2} = \frac{\left(x^2 + y^2 + z^2\right).1 - x(2x)}{\left(x^2 + y^2 + z^2\right)^2} = \frac{x^2 + z^2 - x^2}{\left(x^2 + y^2 + z^2\right)^2}$$

Similarly $\dfrac{\partial^2 V}{\partial y^2} = \dfrac{x^2+z^2-y^2}{\left(x^2+y^2+z^2\right)^2}$ and $\dfrac{\partial^2 V}{\partial z^2}\ \dfrac{x^2+z^2-z^2}{\left(x^2+y^2+z^2\right)^2}$

$$\therefore \quad \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2} = \frac{x^2+z^2+z^2}{\left(x^2+y^2+z^2\right)^2} = \frac{1}{x^2+z^2+z^2}$$

$$\therefore \quad \frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2} = \frac{1}{r^2}$$

Example 16. If $V = \log \dfrac{x^3+y^3}{x^2+y^2}$, show that

$$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}=1$$ (NEHU 2015)

Solution: Given $V = \log \dfrac{x^3+y^3}{x^2+y^2}$

or $$e^V = \frac{x^3+y^3}{x^2+y^2} = \frac{x^3\left[1+\left(y/x\right)^3\right]}{x^2\left[1+\left(y/x\right)^2\right]}$$

or $$e^V = x\,\phi\left(\frac{y}{x}\right)$$

Hence e^V is a homogeneous function in x and y of degree 1

∴ By Euler's Theorem

$$x\frac{\partial}{\partial x}e^V+y\frac{\partial}{\partial y}e^V=1.e^V$$

or $$x\frac{\partial}{\partial V}e^V\frac{\partial V}{\partial x}+y\frac{\partial}{\partial V}e^V\frac{\partial V}{\partial y}=e^V$$

or $$x\,e^V\frac{\partial V}{\partial x}+y\,e^V\frac{\partial V}{\partial y}=e^V$$

or $$x\frac{\partial V}{\partial x}+y\frac{\partial V}{\partial y}=1$$

Example 17. Applying Euler's Theorem to the function $u(x, y) = \tan^{-1}\left[\frac{x^{5/2}+y^{5/2}}{\sqrt{x}-\sqrt{y}}\right]$,

show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$$ (NEHU 2016)

Solution: Given $u = \tan^{-1}\left[\frac{x^{5/2}+y^{5/2}}{\sqrt{x}-\sqrt{y}}\right]$

$$\text{or } \tan u = \frac{x^{5/2}+y^{5/2}}{x^{1/2}-y^{1/2}} = \frac{x^{5/2}\left[1+\left(y/x\right)^{5/2}\right]}{x^{1/2}\left[1-\left(y/x\right)^{1/2}\right]}$$

$$\text{or } \tan u = x^2\,\phi\left(\frac{y}{x}\right)$$

Hence tanu is a homogeneous function of degree 2 in x and y

∴ By Euler's Theorem

$$x\frac{\partial}{\partial x}\tan u+y\frac{\partial}{\partial y}\tan u=2\tan u$$

or $$x\frac{\partial}{\partial u}\tan u\frac{\partial u}{\partial x}+y\frac{\partial}{\partial u}\tan u\frac{\partial u}{\partial y}=2\tan u$$

or $$x\sec^2 u\frac{\partial u}{\partial x}+y\sec^2 u\frac{\partial u}{\partial y}=2\tan u$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2\tan u}{\sec^2 u}$$

or $$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u$$

Exercises

1. Find f_x and f_y in the following functions;

   (i) $\sin^{-1}\left(\frac{y}{x}\right)$ (ii) xtany + ytanx

   (iii) $\frac{1}{\sqrt{x^2+y^2}}$ (iv) $ax^2 + 2hxy + by^2$

2. Find $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial x\,\partial y}$ and $\frac{\partial^2 u}{\partial y\,\partial x}$ in the following

   (i) x cosy + y cosx (ii) $\log(x^2y + xy^2)$

3. If $\sin^{-1}\left(\frac{y}{x}\right) + \tan^{-1}\left(\frac{y}{x}\right) = u(x, y)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

4. If $u = \frac{1}{\sqrt{x^2+y^2+z^2}}$; $x^2 + y^2 + z^2 \neq 0$, then show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

5. Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ if

   (i) $u = \log(x^2 + y^2)$ (ii) $u = \tan^{-1}\left(\frac{y}{x}\right)$

   (iii) $u = e^x (x\cos y - y\sin y)$

6. If $u = \log(x^3 + y^3 - 3xyz)$, show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{3}{(x+y+z)^2}$

7. If $u = \log(x^2 + y^2 + z^2)$, prove that $x\frac{\partial^2 u}{\partial y\partial z} = y\frac{\partial^2 u}{\partial z\partial x} = z\frac{\partial^2 u}{\partial x\,\partial y}$

8. If $u = \tan^{-1}\frac{x^3+y^3}{x+y}$; show that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$

   $= (1-4\sin^2 u)\sin 2u$

9. If $V = z\sin^{-1}\left(\frac{y}{x}\right)$, then show that $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

10. If $V = \left(x^2+y^2+z^2\right)^{-1/2}$, show that

    (i) $x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = -V$ (ii) $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$

11. Verify Euler's Therorem for the following functions

(i) $u = ax^2 + 2hxy + by^2$ (ii) $u = \frac{x-y}{x+y}$

(iii) $u = \sin\left(\frac{x^2+y^2}{xy}\right)$ (iv) $u = x^3 \log\left(\frac{y}{x}\right)$

12. If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\tan u$

13. If $u = \cos^{-1}\left(\frac{x-y}{x+y}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$

14. If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$

15. If u is a homogeneous function of degree n, show that

(i) $x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\,\partial y} = (n-1)\frac{\partial u}{\partial x}$ (ii) $x\frac{\partial^2 u}{\partial x\,\partial y} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}$

and hence deduce that $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$

16. If V = f(u), u being a homogeneous function of degree n in x and y, show that $x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nu\frac{\partial v}{\partial u}$

17. If $u = x\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$; show that

(i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\phi\left(y/x\right)$

(ii) $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$

18. If u is a homogeneous function of x and y of dimensions n the prove that

$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)u^2 = n(n-1)u$ where

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 = x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2}$$

19. If u = $\cos\left(\frac{y}{x}\right)$ show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$

20. If u = x^y prove that $\frac{\partial^2 u}{\partial x\,\partial y}=\frac{\partial^2 u}{\partial y\,\partial x}$

21. If u = $\log\tan\left(\frac{y}{x}\right)$, prove that $\frac{\partial^2 u}{\partial x\,\partial y}=\frac{\partial^2 u}{\partial y\,\partial x}$

22. If u = xlogy, prove that $\frac{\partial^2 u}{\partial x\,\partial y}=\frac{\partial^2 u}{\partial y\,\partial x}$

23. If u = $\cos^{-1}\left(\frac{x}{y}\right)+\cot^{-1}\left(\frac{y}{x}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0$

24. If u = $\tan^{-1}\left(\frac{y}{x}\right)$ find the value of $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$

25. If v = logr and $r^2 = x^2 + y + z^2$, prove that $\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=\frac{1}{r^2}$

26. If u = $\log\left(\frac{x^3+y^3}{x^2+y^2}\right)$ show by Euler's Theorem that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=1$

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ISBN: 978 93 88881 25 8

9 789388 881258

2020 ₹ 425.00

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EBH Publishers (India)
an imprint of Eastern Book House®
136, M.L.N. Road, Panbazar, Guwahati-781001 (India)